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# Spectral Inclusions for Quasi-Fredholm and Saphar Spectra for Integrated Semigroups

## Inclusiones espectrales para Quasi-Fredholm y espectro Saphar de semigrupos integrados

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Abstract. In this paper, we show a spectral inclusion of integrated semigroups for Saphar, essentially Saphar and quasi-Fredholm spectra.

*Key words and phrases.* Integrated Semigroup, Quasi-Fredholm Operator, Saphar Spectrum.

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Resumen. En este trabajo mostramos inclusiones espectrales de semigrupos integrados para el espectro de Saphar, el espectro esencial de Saphar y el espectro cuasi-Fredholm.

*Palabras y frases clave.* semigrupo integrado, operador cuasi-Fredholm, espectro de Saphar.

### 1. Introduction and Preliminaries

Throughout this paper, $X$ denotes a complex Banach space, $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on $X$. Let $A$ be a closed linear operator on $X$ with domain $D(A)$. We denote by $A^*$, $R(A)$, $N(A)$, $R^\infty(A) = \bigcap_{n\geq 0} R(A^n)$ and $\sigma(A)$, respectively the adjoint, the range, the null space, the hyper-range and the spectrum of $A$.

Recall that a closed operator $A$ is said to be a Kato operator or semi-regular if $R(A)$ is closed and $N(A) \subseteq R^\infty(A)$. Denote by $\rho_K(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is Kato}\}$ the Kato resolvent and $\sigma_K(A) = \mathbb{C} \backslash \rho_K(A)$ the Kato spectrum of $A$. It is well known that $\rho_K(A)$ is an open subset of $\mathbb{C}$, see [6, Theorem 2.10].

For subspaces $M$, $N$ of $X$ we write $M \subseteq^e N$ ($M$ is essentially contained in $N$) if there exists a finite-dimensional subspace $F \subset X$ such that $M \subseteq N + F$.

A closed operator $S$ is called a generalized inverse (pseudo inverse) of $A$ if $R(A) \subseteq D(S)$, $R(S) \subseteq D(A)$, $ASA = A$ on $D(A)$ and $SAS = S$ on $D(S)$, see [5, Definition 1.1].

A closed operator $A$ is called a Saphar operator if $A$ has a generalized inverse and $N(A) \subseteq R^\infty(A)$.

If we assume in the definition above that $N(A) \subseteq^e R^\infty(A)$, $A$ is said to be an essentially Saphar operator. The Saphar and essentially Saphar spectra are defined by

$$\sigma_{Sap}(A) = \{\lambda \in \mathbb{C} : \; A - \lambda I \text{ is not Saphar}\} \text{ and}$$

$$\sigma_{Sap}^e(A) = \{\lambda \in \mathbb{C} : \; A - \lambda I \text{ is not essentially Saphar}\}$$

respectively. Integrated semigroups were first defined by Arendt [1]. He showed that certain natural classes of operators, such as adjoint semigroups of $C_0$ semigroups on non-reflexive Banach spaces, give rise to integrated semigroups which are not integrals of $C_0$ semigroups.

A family of bounded linear operators $(S(t))_{t\geq 0}$, on a Banach space $X$ is called an integrated semigroup (once integrated semigroup) iff

(1) $S(0) = 0$;

(2) $S(t)$ is strongly continuous in $t \geq 0$;

(3) $S(r)S(t) = \int_0^r (S(\tau + t) - S(\tau))d\tau = S(t)S(r)$.

In the general setting, a strongly continuous family $(S(t))_{t\geq 0} \subseteq \mathcal{B}(X)$ is called an $n$-times integrated semigroup if (1) and (2) are satisfied and for all $s, t \geq 0$

$$S(r)S(t) = \frac{1}{(n-1)!}\left(\int_0^r (r-\tau)^{n-1} S(\tau + t) - (r + t - \tau)^{n-1} S(\tau) d\tau\right).$$

The differentation spaces $C^n$, $n \geq 0$, are defined by $C^0 = X$ and

$$C^n = \{x \in X : S(\cdot)x \in C^n(\mathbb{R}^+; X)\}.$$

Using this notion, (3) can equivalently be formulated by, for all $x \in X$,

$$S(t)x \in C^1 \text{ and } S'(r)S(t)x = S(r+t)x - S(r)x, \quad \forall r, t \geq 0.$$

The set $N = \{x \in X : S(t)x = 0, \forall t \geq 0\}$ is called the degeneration space of the integrated semigroup $(S(t))_{t\geq 0}$. Moreover $(S(t))_{t\geq 0}$ is called non-degenerate if $N = \{0\}$ and degenerate otherwise.

The generator $A : D(A) \subseteq X \to X$ of a non-degenerate integrated semigroup $(S(t))_{t\geq 0}$ is defined as follows: $x \in D(A)$ and $Ax = y$ iff $x \in C^1$ and $S'(t)x - x = S(t)y$ for all $t \geq 0$. Also, we have: $C^2 \subseteq D(A) \subseteq C^1$ and $Ax = S''(0)x$ for all $x \in C^2$. Moreover $AC^2 \subseteq C^1$ [9, Lemma 3.2].

Note that $A$ is a closed linear operator [9, Lemma 3.3], $S(t) : C^1 \to C^2 \subseteq D(A)$ and $AS(t)x = S''(0)S(t)x = S'(t)x - x$ for all $x \in C^1$. Further $AS(t)x = S(t)Ax$ for all $x \in D(A)$, see [9, Lemma 3.4]. $\int_0^t S(r)dr$ maps $X$ into $D(A)$ and $A\int_0^t S(r)xdr = S(t)x - tx$. A non-degenerate integrated semigroup is uniquely determined by its generator. Let $u : [0,T) \to X$ be continuous such that $\int_0^t u(s)ds \in D(A)$ and $A(\int_0^t u(s)ds) = u(t)$, for all $0 \leq t \leq T$. Then $u = 0$ in $[0,T)$. Arendt [1] showed that if $A$ generates $S_t$ as an $n$-times integrated semigroup, then the Abstract Cauchy Problem $u'(t) = Au(t), u(0) = x$ has a classical solution for all $x \in D(A^{n+1})$.

In order to understand the behavior of the solutions in terms of the data concerning $A$, one seeks information about the spectrum of $S(t)$ in terms of the spectrum of $A$. Unfortunately the spectral mapping theorem $e^{t\sigma_*(A)} = \sigma_*(S(t)) \setminus \{0\}$ often fails, sometimes in dramatic ways, when $S(t)$ is a strongly continuous semigroup. However, the inclusion

$$e^{t\sigma_*(A)} \subseteq \sigma_*(S(t)) \setminus \{0\}$$

always holds, where $\sigma_* \in \{\sigma, \sigma_{sap}, \sigma^e_{sap}\}$ and $S(t)$ is a strongly continuous semigroup, see [8, Theorems 2.1 and 3.2] and [3, Page 276].

The spectral inclusions for various reduced spectra of an $n$-times uniformly exponentially bounded integrated semigroup were studied by Day [2], when $n > 0$. Precisely, he showed the following spectral mapping theorem

$$\sigma_*(S(t)) \cup \{0\} = \left\{ \int_0^t e^{\lambda s} \frac{(t-s)^\alpha}{\Gamma(\alpha)} ds; \lambda \in \sigma_*(S(t)) \right\} \cup \{0\},$$

where $\sigma_* \in \{\sigma_p, \sigma_{ap}\}$ the point spectrum and the approximate point spectrum, $\alpha > 0$ and $\Gamma$ is the Euler integral, see [2, Theorem 3.9]. By combining these results from Day [2, Theorem 3.9] we can conclude that the spectral mapping theorem also holds for the entire spectrum $\sigma(S(t))$, i.e.,

$$\sigma(S(t)) \cup \{0\} = \left\{ \int_0^t e^{\lambda s} \frac{(t-s)^\alpha}{\Gamma(\alpha)} ds; \lambda \in \sigma(S(t)) \right\} \cup \{0\}.$$

Then, in the case of a once uniformly exponentially bounded integrated semigroup, we have

$$\int_0^t e^{s\sigma(A)} ds \cup \{0\} = \sigma(S(t)) \cup \{0\}.$$

According to Lemma 2.1, the inclusion $\int_0^t e^{s\sigma(A)}ds \subset \sigma(S(t))$ holds when $(S(t))_{t\geq 0}$ is an integrated semigroup.

In this work, we will continue in this direction, we will establish the relationship between the spectra of the integrated semigroup and its generator, more precisely, we show that

$$\int_0^t e^{s\sigma_*(A)}ds \subseteq \sigma_*(S(t))$$

where $\sigma_*$ runs through the Saphar, essential Saphar and quasi-Fredholm spectra.

## 2. Spectral Inclusions For Saphar Spectrum

We start with some lemmas which will be needed in the sequel.

**Lemma 2.1.** *Let $A$ be the generator of a non-degenerate integrated semigroup $(S(t))_{t\geq 0}$, $D_\lambda(t)x = \int_0^t e^{\lambda(t-s)}S(s)xds$. Then, for all $\lambda \in \mathbb{C}$, $t \geq 0$, and $n \in \mathbb{N}$,*

*(1)* $(\int_0^t e^{\lambda s}ds - S(t))x = (\lambda - A)D_\lambda(t)x, \forall x \in X$;

*(2)* $(\int_0^t e^{\lambda s}ds - S(t))x = D_\lambda(t)(\lambda - A)x, \forall x \in D(A)$;

*(3)* $N((\lambda - A)^n) \subseteq N(\int_0^t e^{\lambda s}ds - S(t))^n$;

*(4)* $R(\int_0^t e^{\lambda s}ds - S(t))^n \subseteq R(\lambda - A)^n$.

***Proof.*** (1) for all $r$, $t \in [0, +\infty[$ and $x \in X$ we have,

$$\begin{aligned} S(r)D_\lambda(t)x &= S(r)\int_0^t e^{\lambda(t-s)}S(s)xds \\ &= \int_0^t e^{\lambda(t-s)}S(r)S(s)xds \\ &= \int_0^t \int_0^r e^{\lambda(t-s)}[S(\tau+s) - S(\tau)]xd\tau ds \\ &= \int_0^r \int_0^t e^{\lambda(t-s)}[S(\tau+s) - S(\tau)]xdsd\tau. \end{aligned}$$

Then, for all $x \in X$, $D_\lambda(t)x \in C^1$. Furthermore,

$$\begin{aligned}\frac{d}{dr}S(r)D_\lambda(t)x &= \int_0^t e^{\lambda(t-s)}[S(r+s)-S(r)]xds \\ &= \int_0^t e^{\lambda(t-s)}[S(r+s)-S(s)]xds + \int_0^t e^{\lambda(t-s)}S(s)xds \\ &- \int_0^t e^{\lambda(t-s)}S(r)xds \\ &= \int_0^t e^{\lambda(t-s)}\frac{d}{ds}[S(s)S(r)]xds - S(r)\int_0^t e^{\lambda s}xds + D_\lambda(t)x \\ &= S(r)S(t)x + \lambda S(r)D_\lambda(t)x - S(r)\int_0^t e^{\lambda s}xds + D_\lambda(t)x \\ &= S(r)\left(S(t)+\lambda D_\lambda(t) - \int_0^t e^{\lambda s}ds\right)x + D_\lambda(t)x.\end{aligned}$$

Therefore $D_\lambda(t)x \in D(A)$ and $AD_\lambda(t)x = S(t)x + \lambda D_\lambda(t)x - \int_0^t e^{\lambda s}xds$. Thus,

$$(\int_0^t e^{\lambda s}ds - S(t))x = (\lambda - A)D_\lambda(t)x.$$

(2) Let $x \in D(A)$, then

$$\begin{aligned}D_\lambda(t)Ax &= \int_0^t e^{\lambda(t-s)}S(s)Axds \\ &= \int_0^t e^{\lambda(t-s)}(S'(s)x - x)ds \\ &= \int_0^t e^{\lambda(t-s)}S'(s)xds - \int_0^t e^{\lambda s}xds \\ &= S(s)x + \lambda D_\lambda(t)x - \int_0^t e^{\lambda s}xds.\end{aligned}$$

Hence

$$(\int_0^t e^{\lambda s}ds - S(t))x = D_\lambda(t)(\lambda - A)x.$$

The assertions (3) and (4) are consequences of the statements (1) and (2). ☑

**Lemma 2.2.** *Let $(S(t))_{t\geq 0}$ be a non-degenerate integrated semigroup on $X$ with generator $A$. For $\lambda \in \mathbb{C}$ and $t \geq 0$, let $L_\lambda(t)x = \int_0^t e^{-\lambda s}D_\lambda(s)xds$. Then*

*(1) $L_\lambda(t)$ is a bounded linear operator on $X$,*

*(2)* $\forall x \in X$, $L_\lambda(t)x \in D(A)$ *and* $(\lambda - A)L_\lambda(t) + G_\lambda(t)D_\lambda(t) = \phi_\lambda(t)I$ *with* $G_\lambda(t) = e^{-\lambda t}I$ *and* $\phi_\lambda(t) = \int_0^t \int_0^\tau e^{-\lambda\sigma} d\sigma d\tau$,

*(3) The operators* $L_\lambda(t)$, $G_\lambda(t)$, $D_\lambda(t)$ *and* $(\lambda - A)$ *are pairwise commuting.*

***Proof***. (1) Obvious.

(2) For all $r \geq 0$ and $x \in X$, we have

$$\begin{aligned} S(r)L_\lambda(t)x &= \int_0^t e^{-\lambda\tau} S(r)D_\lambda(\tau)x d\tau \\ &= \int_0^t \int_0^\tau e^{-\lambda\sigma} \int_0^r [S(u+\sigma) - S(u)]x du d\sigma d\tau \\ &= \int_0^t \int_0^\tau \int_0^r e^{-\lambda\sigma} [S(u+\sigma) - S(u)]x du d\sigma d\tau \\ &= \int_0^r \int_0^t \int_0^\tau e^{-\lambda\sigma} [S(u+\sigma) - S(u)]x d\sigma d\tau du. \end{aligned}$$

Therefore, $L_\lambda(t)x \in C^1$ and

$$\begin{aligned} \frac{d}{dr} S(r)L_\lambda(t)x &= \int_0^t \int_0^\tau e^{-\lambda\sigma} [S(r+\sigma) - S(r)]x d\sigma d\tau \\ &= \int_0^t \int_0^\tau e^{-\lambda\sigma} [S(r+\sigma) - S(\sigma)]x d\sigma d\tau + L_\lambda(t)x - \phi_\lambda(t)S(r)x \\ &= \int_0^t \int_0^\tau e^{-\lambda\sigma} \frac{d}{d\sigma} S(\sigma)S(r)x d\sigma d\tau + L_\lambda(t)x - \phi_\lambda(t)S(r)x \\ &= S(r)[e^{-\lambda t}D_\lambda(t)x + \lambda L_\lambda(t)x - \phi_\lambda(t)x] + L_\lambda(t)x. \end{aligned}$$

Therefore, $AL_\lambda(t)x = e^{-\lambda t}D_\lambda(t)x + \lambda L_\lambda(t)x - \phi_\lambda(t)x$.
So, we have $(\lambda - A)L_\lambda(t) + G_\lambda(t)D_\lambda(t) = \phi_\lambda(t)I$ with $G_\lambda(t) = e^{-\lambda t}I$.

(3) For all $t \geq 0$, $L_\lambda(t)$ and $D_\lambda(t)$ are commuting. Indeed, for all $t, s \geq 0$ we have,

$$\begin{aligned} D_\lambda(t)D_\lambda(s)x &= \int_0^t e^{\lambda(t-u)} S(u)D_\lambda(s)x du \\ &= \int_0^t e^{\lambda(t-u)} S(u) \int_0^s e^{\lambda(s-v)} S(v)x dv du \\ &= \int_0^t \int_0^s e^{\lambda(t-u)} e^{\lambda(s-v)} S(u)S(v)x dv du \\ &= \int_0^s e^{\lambda(s-v)} S(v) \int_0^t e^{\lambda(t-u)} S(u)x du dv \\ &= D_\lambda(s)D_\lambda(t)x. \end{aligned}$$

Therefore,

$$\begin{aligned} L_\lambda(t)D_\lambda(t)x &= \int_0^t e^{-\lambda u} D_\lambda(u)D_\lambda(t)xdu \\ &= \int_0^t e^{-\lambda u} D_\lambda(t)D_\lambda(u)xdu \\ &= D_\lambda(t)\int_0^t e^{-\lambda u} D_\lambda(u)xdu \\ &= D_\lambda(t)L_\lambda(t)x \end{aligned}$$

For all $x \in D(A)$, we have

$$\begin{aligned} L_\lambda(t)(\lambda - A)x &= \int_0^t e^{-\lambda s} D_\lambda(s)(\lambda - A)xds \\ &= \int_0^t e^{-\lambda s}\left(\int_0^s e^{\lambda r}dr - S(s)\right)xds \\ &= \phi_\lambda(t)x - \int_0^t e^{-\lambda s}S(s)xds \\ &= \phi_\lambda(t)x - G_\lambda(t)D_\lambda(t)x \\ &= (\lambda - A)L_\lambda(t)x. \end{aligned}$$

It is easy to see that $(\lambda - A)G_\lambda(t)x = G_\lambda(t)(\lambda - A)x$, for all $x \in D(A)$.

Also, by lemma 2.1 $(\lambda - A)D_\lambda(t)x = D_\lambda(t)(\lambda - A)x$, for all $x \in D(A)$.

☑

**Lemma 2.3.** *Let $(S(t))_{t\geq 0}$ be a non-degenerate integrated semigroup on $X$ with generator $A$. For all $n \in \mathbb{N}^*$, $\lambda \in \mathbb{C}$ and $t > 0$, there exist two operators $F_n(t), H_n(t) \in \mathcal{B}(X)$ such that,*

*(1)* $\forall x \in X$, $F_n(t)x \in D(A^n)$ *and* $(\lambda - A)^n F_n(t) + H_n(t)D^n_\lambda(t) = I$,

*(2)* $(\lambda - A)^n$, $F_n(t)$, $H_n(t)$ *and* $D^n_\lambda(t)$ *are pairwise commuting.*

***Proof***. By lemma 2.2, there exist two operators $L_\lambda(t)$ and $G_\lambda(t)$ such that $(\lambda - A)L_\lambda(t) + G_\lambda(t)D_\lambda(t) = I$. For all $n \geq 1$ and $x \in X$, we have $L^n_\lambda(t)x \in D(A^n)$. In fact, the proof is by induction. For $n = 1$, from lemma 2.2 $L_\lambda(t)x \in D(A)$. suppose that $L^{n-1}_\lambda(t)x \in D(A^{n-1})$, so $L^n_\lambda(t)x \in D(A^{n-1})$ and

$$\begin{aligned} (\lambda - A)^{n-1}L^n_\lambda(t)x &= [(\lambda - A)L_\lambda(t)]^{n-1}L_\lambda(t)x \\ &= L_\lambda(t)[(\lambda - A)L_\lambda(t)]^{n-1}x \in D(A), \end{aligned}$$

hence, $L_\lambda^n(t)x \in D(A^n)$. Furthermore,

$$\begin{aligned}(\lambda - A)^n L_\lambda^n(t) &= [(\lambda - A)L_\lambda(t)]^n \\ &= [I - G_\lambda(t)D_\lambda(t)]^n \\ &= I - L_{1,n}(t)D_\lambda(t),\end{aligned}$$

with $L_{1,n}(t) = \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}G_\lambda^k(t)D_\lambda^{k-1}(t)$.

Therefore $(\lambda - A)^n L_\lambda^n(t) + L_{1,n}(t)D_\lambda(t) = I$. Similarly, we have

$$\begin{aligned}L_{1,n}^n(t)D_\lambda^n(t) &= [I - (\lambda - A)^n L_\lambda^n(t)]^n \\ &= I - (\lambda - A)^n \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}(\lambda - A)^{n(k-1)}L_\lambda^{nk}(t).\end{aligned}$$

We define $F_n(t) = \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}(\lambda - A)^{n(k-1)}L_\lambda^{nk}(t)$ and $H_n(t) = L_{1,n}^n(t)$. Then $(\lambda - A)^n F_n(t) + H_n(t)D_\lambda^n(t) = I$. Moreover the operators $(\lambda - A)^n$, $F_n(t)$, $H_n(t)$ and $D_\lambda^n(t)$ are pairwise commuting. ☑

**Lemma 2.4.** *Let $(S(t))_{t\geq 0}$ be a non-degenerate integrated semigroup with generator $A$. Then for all $t > 0$ we have,*

*$\int_0^t e^{\lambda s}ds - S(t)$ has a generalized inverse $\Longrightarrow$ $\lambda - A$ has a generalized inverse.*

***Proof.*** Suppose that $\int_0^t e^{\lambda s}ds - S(t)$ has a generalized inverse. Since $\int_0^t e^{\lambda s}ds - S(t)$ is a bounded linear operator, by [7, Proposition 1, Chapter I.13] there exists $R \in \mathcal{B}(X)$ such that,

$$\left(\int_0^t e^{\lambda s}ds - S(t)\right) R \left(\int_0^t e^{\lambda s}ds - S(t)\right) = \int_0^t e^{\lambda s}ds - S(t).$$

According to Lemma 2.2, we have $(\lambda - A)L_\lambda(t) + G_\lambda(t)D_\lambda(t) = \phi_\lambda(t)I$, then

$$\begin{aligned}\phi_\lambda(t)(\lambda - A) &= (\lambda - A)F_\lambda(t)(\lambda - A) + D_\lambda(t)G_\lambda(t)(\lambda - A) \\ &= (\lambda - A)F_\lambda(t)(\lambda - A) + (\lambda - A)D_\lambda(t)G_\lambda(t) \\ &= (\lambda - A)F_\lambda(t)(\lambda - A) + \left(\int_0^t e^{\lambda s}ds - S(t)\right)G_\lambda(t) \\ &= (\lambda - A)F_\lambda(t)(\lambda - A) + \\ &\quad \left(\int_0^t e^{\lambda s}ds - S(t)\right) R \left(\int_0^t e^{\lambda s}ds - S(t)\right)G_\lambda(t) \\ &= (\lambda - A)F_\lambda(t)(\lambda - A) + (\lambda - A)D_\lambda(t)R(\lambda - A)D_\lambda(t)G_\lambda(t) \\ &= (\lambda - A)F_\lambda(t)(\lambda - A) + (\lambda - A)D_\lambda(t)RD_\lambda(t)G_\lambda(t)(\lambda - A) \\ &= (\lambda - A)[F_\lambda(t) + D_\lambda(t)RD_\lambda(t)G_\lambda(t)](\lambda - A).\end{aligned}$$

Let $H_\lambda = \frac{1}{\phi_\lambda(t)}\left[L_\lambda(t) + D_\lambda(t)RD_\lambda(t)G_\lambda(t)\right]$. Then $H_\lambda$ is a bounded linear operator, $R(H_\lambda) \subseteq D(A)$ and $(\lambda - A)H_\lambda(\lambda - A) = \lambda - A$. Let $K_\lambda = H_\lambda(\lambda - A)H_\lambda$. It follows from Lemma 2.1 and Lemma 2.2 that $K_\lambda$ is a bounded linear operator and $R(K_\lambda) \subseteq D(A)$. Moreover, we have

$$\begin{aligned}(\lambda - A)K_\lambda(\lambda - A) &= (\lambda - A)H_\lambda(\lambda - A)H_\lambda(\lambda - A)\\ &= (\lambda - A)H_\lambda(\lambda - A)\\ &= \lambda - A\end{aligned}$$

and

$$\begin{aligned}K_\lambda(\lambda - A)K_\lambda &= H_\lambda(\lambda - A)H_\lambda(\lambda - A)H_\lambda\\ &= H_\lambda(\lambda - A)H_\lambda\\ &= K_\lambda.\end{aligned}$$

Hence $\lambda - A$ has a generalized inverse. ☑

**Theorem 2.5.** *Let $(S(t))_{t\geq 0}$ be a non-degenerate integrated semigroup with generator $A$. Then for all $t > 0$,*

$$\int_0^t e^{s\sigma_{Sap}(A)}ds \subseteq \sigma_{Sap}(S(t)) \quad \text{and} \quad \int_0^t e^{s\sigma^e_{Sap}(A)}ds \subseteq \sigma^e_{Sap}(S(t)).$$

***Proof***. Assume that $\int_0^t e^{\lambda s}ds - S(t)$ is a Saphar operator, then $\int_0^t e^{\lambda s}ds - S(t)$ has a generalized inverse and $N\left(\int_0^t e^{\lambda s}ds - S(t)\right) \subseteq R^\infty\left(\int_0^t e^{\lambda s}ds - S(t)\right)$. By Lemma 2.4, $\lambda - A$ has a generalized inverse, and we have:

$$N(\lambda - A) \subseteq N\left(\int_0^t e^{\lambda s}ds - S(t)\right) \subseteq R^\infty\left(\int_0^t e^{\lambda s}ds - S(t)\right) \subseteq R^\infty(\lambda - A).$$

Therefore $\lambda - A$ is a Saphar operator.

Let $M$ a finite dimensional subspace of $X$. We have,

$N(\lambda - A) \subseteq N\left(\int_0^t e^{\lambda s}ds - S(t)\right) \subseteq R^\infty\left(\int_0^t e^{\lambda s}ds - S(t)\right) + M \subseteq R^\infty(\lambda - A) + M$. Hence $\int_0^t e^{\lambda s}ds - S(t)$ is essentially Saphar implies that $\lambda - A$ is so. ☑

## 3. Spectral Inclusion For Quasi-Fredholm Spectrum

We recall from [4] some definitions:

**Definition 3.1.** Let $T$ be a closed linear operator on $X$ and let

$$\Delta(T) = \{n \in \mathbb{N} : \forall m \geq n, R(T^n) \cap N(T) = R(T^m) \cap N(T)\}.$$

The degree of stable iteration $dis(T)$ of $T$ is defined as $dis(T) = \inf \Delta(T)$ with $dis(T) = \infty$ if $\Delta(T) = \varnothing$.

**Definition 3.2.** Let $T$ be a closed linear operator on $X$. $T$ is called a quasi-Fredholm operator of degree $d$ if there exists an integer $d \in \mathbb{N}$ such that

(1) $dis(T) = d$;

(2) $R(T^n)$ is closed in $X$ for all $n \geq d$;

(3) $R(T) + N(T^n)$ is closed in $X$ for all $n \geq d$.

The quasi-Fredholm spectrum is defined by

$$\sigma_{qF}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a quasi-Fredholm}\}.$$

**Proposition 3.3.** *Let $(S(t))_{t\geq 0}$ be a non-degenerate integrated semigroup with generator $A$. Then*

$$dis(\lambda - A) \leq dis(\int_0^t e^{\lambda s} ds - S(t)).$$

***Proof*.** If $dis(\int_0^t e^{\lambda s} ds - S(t)) = +\infty$. In this case, the result is obvious.

If $dis(\int_0^t e^{\lambda s} ds - S(t)) = d \in \mathbb{N}\backslash\{0\}$. Then for all $n \geq d$,

$$R\left(\int_0^t e^{\lambda s} ds - S(t)\right)^n \bigcap N\left(\int_0^t e^{\lambda s} ds - S(t)\right) =$$

$$R\left(\int_0^t e^{\lambda s} ds - S(t)\right)^d \bigcap N\left(\int_0^t e^{\lambda s} ds - S(t)\right).$$

We show that for all $n \geq d$,

$$R(\lambda - A)^n \cap N(\lambda - A) = R(\lambda - A)^d \cap N(\lambda - A).$$

Let $y \in R(\lambda - A)^d \cap N(\lambda - A)$, then there exists $x \in D(A^d)$ such that $y = (\lambda - A)^d x$. Then, according to lemma 2.3, there exist two operators $F_d(t)$ and $G_d(t)$ such that

$$(\lambda - A)^d F_d(t) + D_\lambda^d(t) G_d(t) = I.$$

Therefore,

$$\begin{aligned} y &= (\lambda - A)^d F_d(t) y + D_\lambda^d(t) G_d(t) y \\ &= (\lambda - A)^{d-1} F_d(t)(\lambda - A) y + \left(\int_0^t e^{\lambda s} ds - S(t)\right)^d G_d(t) x \\ &= \left(\int_0^t e^{\lambda s} ds - S(t)\right)^d G_d(t) x. \end{aligned}$$

Therefore,

$y \in R\left(\int_0^t e^{\lambda s}ds - S(t)\right)^d \cap N\left(\int_0^t e^{\lambda s}ds - S(t)\right) \subseteq R\left(\int_0^t e^{\lambda s}ds - S(t)\right)^n \subseteq R(\lambda - A)^n$. Consequently $y \in R(\lambda - A)^n \cap N(\lambda - A)$.

If $d = 0$, for all $n \geq d$, we have

$R(\int_0^t e^{\lambda s}ds - S(t))^n \cap N(\int_0^t e^{\lambda s}ds - S(t)) = N(\int_0^t e^{\lambda s}ds - S(t))$. Consequently $N(\int_0^t e^{\lambda s}ds - S(t)) \subseteq R(\int_0^t e^{\lambda s}ds - S(t))^n$. By Lemma 2.1, we have $N(\lambda - A) \subseteq N(\int_0^t e^{\lambda s}ds - S(t)) \subseteq R(\int_0^t e^{\lambda s}ds - S(t))^n \subseteq R(\lambda - A)^n$. Hence $N(\lambda - A) \cap R(\lambda - A)^n = N(\lambda - A) \cap R(\lambda - A)^0$ and so $dis(\lambda - A) = 0$. ☑

**Proposition 3.4.** *Let $(S(t))_{t \geq 0}$ be a non-degenerate integrated semigroup on $X$ with generator $A$.*

*If $R(\int_0^t e^{\lambda s}ds - S(t))^n$ is closed for all $n \geq d$, then $R(\lambda - A)^n$ is closed for all $n \geq d$.*

***Proof.*** Let $y_p = (\lambda - A)^n x_p$ be a sequence which conveges to $y$. We show that $y \in R(\lambda - A)^n$. According to lemma 2.3 there exist two bounded linear operators $F_n(t)$ and $G_n(t)$ such that

$$(\lambda - A)^n F_n(t) + D_\lambda^n(t) G_n(t) = I. \tag{1}$$

It follows that $D_\lambda^n(t)y_p = D_\lambda^n(t)(\lambda - A)^n x_p = (\int_0^t e^{\lambda s}ds - S(t))^n x_p \in R(\int_0^t e^{\lambda s}ds - S(t))^n$. Since $D_\lambda^n(t)y_p$ converges to $D_\lambda^n(t)y$ and $R(\int_0^t e^{\lambda s}ds - S(t))^n$ is closed, then $D_\lambda^n(t)y \in R(\int_0^t e^{\lambda s}ds - S(t))^n$, so there exists $z \in X$ such that $D_\lambda^n(t)y = (\int_0^t e^{\lambda s}ds - S(t))^n z$. By (1) we have $(\lambda - A)^n F_n(t)y_p + G_n(t)D_\lambda^n(t)y_p = y_p$. Going to the limit, we obtain

$$\begin{aligned} y &= (\lambda - A)^n F_n(t)y + \left(\int_0^t e^{\lambda s}ds - S(t)\right)^n G_n(t)z \\ &= (\lambda - A)^n [F_n(t)y + D_\lambda^n(t)G_n(t)z] \in R(\lambda - A)^n. \end{aligned}$$

Hence, $R(\lambda - A)^n$ is closed for all $n \geq d$. ☑

**Proposition 3.5.** *Let $(S(t))_{t \geq 0}$ be a non-degenerate integrated semigroup on $X$ with generator $A$ and $d \in \mathbb{N}\backslash\{0\}$. If $R(\int_0^t e^{\lambda s}ds - S(t)) + N(\int_0^t e^{\lambda s}ds - S(t))^d$ is closed in $X$, then $R(\lambda - A) + N(\lambda - A)^d$ is closed.*

***Proof.*** Suppose that $R(\int_0^t e^{\lambda s}ds - S(t)) + N(\int_0^t e^{\lambda s}ds - S(t))^d$ is closed in $X$. Let $y_n = (\lambda - A)x_n + z_n$ be a sequence which converges to $y$, with $x_n \in X$ and $z_n \in N(\lambda - A)^d$. As $D_\lambda^d(t)y_n = D_\lambda^d(t)(\lambda - A)x_n + D_\lambda^d(t)z_n \in R(\int_0^t e^{\lambda s}ds - S(t)) + N(\int_0^t e^{\lambda s}ds - S(t))^d$, then $D_\lambda^d(t)y \in R(\int_0^t e^{\lambda s}ds - S(t)) + N(\int_0^t e^{\lambda s}ds - S(t))^d$. There exist $x \in X$ and $z \in N(\int_0^t e^{\lambda s}ds - S(t))^d$ such that $D_\lambda^d(t)y =$

$(\int_0^t e^{\lambda s}ds - S(t))x + z$. So $D_\lambda^{2d}(t)y = D_\lambda^d(t)(\int_0^t e^{\lambda s}ds - S(t))x + D_\lambda^d(t)z$ where $D_\lambda^d(t)z \in N(\lambda - A)^d$, which implies that

$$\begin{aligned} y &= (\lambda - A)^d F_d(t)y + D_\lambda^{2d}(t)G_d(t)y \\ &= (\lambda - A)^d F_d(t)y + G_d(t)D_\lambda^d(t)\left(\int_0^t e^{\lambda s}ds - S(t)\right)x + G_d(t)D_\lambda^d(t)z \\ &= (\lambda - A)\left((\lambda - A)^{d-1}F_d(t)y + G_d(t)D_\lambda^d(t)\right) + G_d(t)D_\lambda^d(t)z. \end{aligned}$$

Therefore, $y \in R(\lambda - A) + N(\lambda - A)^d$. Consequently $R(\lambda - A) + N(\lambda - A)^d$ is closed ☑

**Corollary 3.6.** *Let $(S(t))_{t\geq 0}$ be a non-degenerate integrated semigroup on $X$ with generator $A$. If $R(\int_0^t e^{\lambda s}ds - S(t))$ is closed in $X$, then $R(\lambda - A)$ is closed.*

**Theorem 3.7.** *Let $(S(t))_{t\geq 0}$ be a non-degenerate integrated semigroup on $X$ with generator $A$. Then for all $t > 0$,*

$$\int_0^t e^{s\sigma_{qF}(A)}ds \subseteq \sigma_{qF}(S(t)).$$

***Proof***. This is a direct result of the three last propositions. ☑

***Remark* 3.8.** For all $t > 0$, we have

$$\int_0^t e^{s\sigma_K(A)}ds \subseteq \sigma_K(S(t)).$$

Indeed, if $\int_0^t e^{\lambda s}ds - S(t)$ is Kato, then $R(\int_0^t e^{\lambda s}ds - S(t))$ is closed and $dis(\int_0^t e^{\lambda s}ds - S(t)) = 0$. According to proposition 3.3 and corollary 3.6, we have that $R(\lambda - A)$ is closed and $dis(\lambda - A) = 0$. Therefore $\lambda - A$ is Kato.

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