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The Gauss decomposition of products of spherical harmonics

Descomposición de Gauss del producto de armónicas esféricas

Ricardo Estrada

Louisiana State University, Baton Rouge, USA

Abstract. The product of two homogeneous harmonic polynomials is homogeneous, but not harmonic, in general. We give formulas for the Gauss decomposition of the product of two homogeneous harmonic polynomials.

Key words and phrases. Harmonic polynomials, Gauss decomposition, products of spherical harmonics.

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Resumen. El producto de dos polinomios armónicos y homogéneos es homogéneo pero no armónico, en general. Damos fórmulas para la descomposición de Gauss del producto de dos polinomios armónicos y homogéneos

Palabras y frases clave. Polinomios armónicos, descomposición de Gauss, producto de armónicas esféricas.

1. Introduction

Let us denote by $\mathcal{H}_m$ the space of homogeneous harmonic polynomials of degree m in n variables. If the elements of $\mathcal{H}_m$ are considered as polynomial functions in $\mathbb{R}^n$ they are called *solid harmonics* of degree m, while when considered as functions on the unit sphere $\mathbb{S}$ they are called *spherical harmonics* of degree m.

Any homogeneous polynomial of degree m, $p \in \mathcal{P}_m$, can be written in a unique way as

$$p(\mathbf{x}) = \sum_{j=0}^{[\![m/2]\!]} r^{2j} \mathsf{W}_{m-2j}(\mathbf{x}) \,, \tag{1}$$

where $\mathsf{W}_{m-2j} \in \mathcal{H}_{m-2j}$ and where as usual $r^2 = |\mathbf{x}|^2 = \sum_{i=1}^n x_i^2$. This is the Gauss decomposition of p. If $\mathsf{Y}_k \in \mathcal{H}_k$ and $\mathsf{Y}'_m \in \mathcal{H}_m$ then $\mathsf{Y}_k \mathsf{Y}'_m \in \mathcal{P}_{m+k}$ but in general it will not belong to $\mathcal{H}_{m+k}$. The aim of this article is to give the Gauss decomposition of this product. Indeed, we prove that

$$\mathsf{Y}_k \mathsf{Y}'_m = \sum_{q=0}^{k} r^{2q} \mathsf{Z}_{m+k-2q}\,, \tag{2}$$

where the $\mathsf{Z}_{m+k-2q} \in \mathcal{H}_{m+k-2q}$ are given as

$$\binom{k}{q} \sum_{j=q}^{k} \binom{k-q}{j-q} B_{j,q+1}^{(m+k-j)} (-1)^{j-q}\, r^{2j-2q} \mathsf{Y}_k \left(\mathbf{x}^{k-j}\nabla^j\right) \mathsf{Y}'_m\,, \tag{3}$$

the $B_{j,q+1}^{(m+k-j)}$ being constants, defined in (48), and where $\mathsf{Y}_k\left(\mathbf{x}^{k-j}\nabla^j\right)$, the notation employed to denote certain differential operators with polynomial coefficients, is explained in Section 2.

Formulas for the Gauss decomposition of the product of spherical harmonics have attracted the attention of researchers since long ago. In the classic book of Hobson [13, Sects 52-53] we find the expression of the product of Legendre polynomials as a sum of Legendre polynomials, which is exactly the Gauss decomposition of the product of *zonal harmonics,* explained in Subsection 3.1, in three variables[1]. The case $n = 2$ is even older, since the Gauss decomposition is nothing but the well known product to sum formulas from elementary trigonometry,

$$\cos k\theta \cos m\theta = \frac{1}{2}\cos(k+m)\theta + \frac{1}{2}\cos(k-m)\theta\,, \tag{4}$$

$$\cos k\theta \sin m\theta = \frac{1}{2}\sin(k+m)\theta + \frac{1}{2}\sin(k-m)\theta\,, \tag{5}$$

$$\sin k\theta \sin m\theta = -\frac{1}{2}\cos(k+m)\theta + \frac{1}{2}\cos(k-m)\theta\,. \tag{6}$$

Interestingly these elementary formulas are the starting point of the method for solving integral equations on a circle as presented in [7] and a similar method for solving equations over a sphere could be constructed from the formulas of the present study.

The plan of the paper is as follows. In Section 2 we explain a rather useful notation employed to manipulate symmetric tensors and particularly the expressions that appear in our analysis. The formulas for $k = 1$ and $k = 2$ are given in Section 3; they already find use in the description of zonal harmonics presented in the Subsection 3.1. The general formula is stated in the Theorem 4.1 of Section 4, where several particular cases are considered. An interesting identity, corollary of the general formula, is given in the next section, while the

[1] Hobson gives credit for the formulas to F. E. Neumann, who gave them in his 1878 book *Beiträge zur Theorie der Kugelfunctionen.*

proof of the theorem is presented in Section 6. Finally in Section 7 we apply our analysis to obtain formulas involving the integrals of the product of three spherical harmonics[2].

Spherical harmonics have been studied in detail for centuries, as one can see in the texts [3, 10, 16], but we would like to point out the recent interest in harmonic polynomials in Mathematical Physics as they play a pivotal role in Stora's fine notion of divergent amplitudes [14, 18], as well as in several aspects of the theory of multipoles [5, 15]. The formulas given in this article are of general interest, but they will be particularly useful in these areas as well as in Fourier analysis, as needed in Mathematical Physics [1, 2], and in integral geometry [9].

2. The symmetric algebra

If E is a vector space, we denote by $\bigvee(E) = \sum_{N=0}^{\infty} \bigvee_N(E)$ the symmetric algebra of E. The space $\bigvee_N(E)$ is the subspace of $\otimes_N E = E \otimes \cdots \otimes E$, N times, consisting of all symmetric tensors.

Notice that there is an operator $\mathfrak{s} : \otimes_N E \to \bigvee_N(E)$, the symmetrization, given as

$$\mathfrak{s}(\mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_N) = \frac{1}{n!} \sum_{\sigma \in S_N} \mathbf{v}_{\sigma(1)} \otimes \cdots \otimes \mathbf{v}_{\sigma(N)}, \tag{7}$$

that is, if $A = \{A_{i_1 \cdots i_N}\}$, then

$$\mathfrak{s}(A)_{i_1 \cdots i_N} = \frac{1}{n!} \sum_{\sigma \in S_N} A_{\sigma(i_1) \cdots \sigma(i_N)}. \tag{8}$$

The symmetric product of the symmetric tensors $T \in \bigvee_N(E)$ and $S \in \bigvee_M(E)$ is given as

$$T \vee S = \mathfrak{s}(T \otimes S). \tag{9}$$

Example 2.1. If $T = \{t_i\}$ and $S = \{s_j\}$ are vectors (tensors of first order) then $T \vee S$ is the symmetric matrix (tensor of second order)

$$(T \vee S)_{ij} = \frac{1}{2}(t_i s_j + t_j s_i). \tag{10}$$

Example 2.2. If $T = \{t_{ij}\}$ while $s = \{s_k\}$ then

$$(T \vee S)_{ijk} = \frac{1}{3}(t_{ij} s_k + t_{jk} s_i + t_{ik} s_j). \tag{11}$$

We denote as $T^{N_\vee}$ the symmetric product of T with itself N times.

[2] It is interesting to observe that, as reported in [13], formulas for the integrals of the product of three Legendre polynomials have been considered for a long time, as they appear, without proof, in the 1877 book *Spherical Harmonics* by Ferrera, a proof being given by J. C. Adams in 1878.

Example 2.3. If $T=\{t_{ij}\}$ while $s=\{s_k\}$ then

$$\left(T\vee S^{2\vee}\right)_{ijkl}=\frac{1}{6}\left(t_{ij}s_ks_l+t_{ik}s_js_l+t_{il}s_js_k+t_{jk}s_is_l+t_{jl}s_is_k+t_{kl}s_is_j\right). \tag{12}$$

Similarly,

$$\left(T^{2\vee}\right)_{ijkl}=\frac{1}{3}\left(t_{ij}t_{jl}+t_{ik}t_{jl}+t_{il}t_{jk}\right). \tag{13}$$

Our next goal is to show how each homogeneous polynomial of degree N in n real variables, $p\in\mathcal{P}_N\left(\mathbb{R}^n\right)$ gives rise to a function in $\bigvee_N\left(\mathbb{R}^n\right)$. Indeed, we can write

$$p\left(x_1,\ldots,x_n\right)=\sum_{i_1=1}^{n}\cdots\sum_{i_N=1}^{n}A^{i_1\cdots i_N}x_{i_1}\cdots x_{i_N}, \tag{14}$$

for some *symmetric*[3] tensor $\left\{A^{i_1\cdots i_N}\right\}$. We can then define $\widetilde{p}:\bigvee_N\left(\mathbb{R}^n\right)\to\mathbb{R}$ (or $\mathbb{C}$) by putting

$$\widetilde{p}\left(\{t_{i_1\cdots i_N}\}\right)=\sum_{i_1=1}^{n}\cdots\sum_{i_N=1}^{n}A^{i_1\cdots i_N}t_{i_1\cdots i_N}. \tag{15}$$

That is, $\widetilde{p}\in\left(\bigvee_N\left(\mathbb{R}^n\right)\right)'$.

Example 2.4. A simple example will clarify the idea. If

$$p\left(x_1,x_2,x_3,x_4,x_5\right)=x_1^2x_2+x_3x_4x_5, \tag{16}$$

and $a=\{a_{ijk}\}_{i,j,k=1}^{5}\in\bigvee_3\left(\mathbb{R}^5\right)$ then

$$\widetilde{p}\left(a\right)=a_{112}+a_{345}. \tag{17}$$

It is important to observe that the polynomial p can be recovered from $\widetilde{p}$ by the simple formula

$$p\left(x\right)=\widetilde{p}\left(x^{N_\vee}\right). \tag{18}$$

From now on, we shall denote $\widetilde{p}$ simply as p since no confusion should arise.

This construction also allow us to construct from a given homogeneous polynomial $p\in\mathcal{P}_N\left(\mathbb{R}^n\right)$ several polynomials in two variables $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$ as follows. Indeed, if $0\leq L\leq N$ then we can consider the polynomial $q_L\left(\mathbf{x},\mathbf{y}\right)=p\left(\mathbf{x}^{L_\vee}\vee\mathbf{y}^{(N-L)_\vee}\right)$, and in particular the differential operators

$$q_L\left(\mathbf{x},\nabla\right)=p\left(\mathbf{x}^{L_\vee}\vee\nabla^{(N-L)_\vee}\right). \tag{19}$$

Notice that $q_0\left(\mathbf{x},\mathbf{y}\right)=p\left(\mathbf{x}\right)$ for all $\mathbf{y}$.

[3]Naturally there are other non-symmetric tensors that also work.

Example 2.5. If $p = x_1^2 x_2$, then $q_0(\mathbf{x}, \nabla) = x_1^2 x_2$,

$$q_1(\mathbf{x}, \nabla) = \frac{1}{3}\left(2x_1 x_2 \nabla_1 + x_1^2 \nabla_2\right), \tag{20}$$

and $q_2(\mathbf{x}, \nabla) = \nabla_1^2 \nabla_2$.

Example 2.6. If $p = x_1^{k-2} x_2^2$ then

$$p\left(\mathbf{x} \vee \nabla^{(k-1)_\vee}\right) = \frac{k-2}{k} x_1 \nabla_1^{k-3} \nabla_2^2 + \frac{2}{k} x_2 \nabla_1^{k-2} \nabla_2, \tag{21}$$

and consequently, if $q = x_1^{k-2} r^2$ then

$$q\left(\mathbf{x} \vee \nabla^{(k-1)_\vee}\right) = \frac{k-2}{k} x_1 \nabla_1^{k-3} \Delta + \frac{2}{k} D \nabla_1^{k-2}, \tag{22}$$

where Δ is the Laplacian and where D is Euler's operator $\sum_{j=1}^n x_j \nabla_j$.

From now on we shall simplify the notation by writing

$$\mathbf{x}^{L_\vee} \vee \nabla^{(N-L)_\vee} \quad \text{as} \quad \mathbf{x}^L \nabla^{N-L}, \tag{23}$$

since no confusion should arise and the formulas can be written in a more compact way. This is the notation employed in [8] when considering the distributional derivatives of power potentials.

3. The first and second order formulas

Let us start by considering the Gauss decomposition of the product $\mathsf{Y}_k \mathsf{Y}'_m$ of two harmonic polynomials when $k = 1$. It is enough to consider the case when $\mathsf{Y}_k = x_i$ for some i. We have the ensuing very simple formula.

Proposition 3.1. *If* $\mathsf{Y}'_m \in \mathcal{H}_m$, $m \geq 1$, *then the Gauss decomposition of* $x_i \mathsf{Y}'_m$ *is*

$$x_i \mathsf{Y}'_m = \mathsf{W}_{m+1} + r^2 \mathsf{W}_{m-1}, \tag{24}$$

where

$$\mathsf{W}_{m+1} = x_i \mathsf{Y}'_m - r^2 \beta_m (\mathsf{Y}'_m)_{,i}, \qquad \mathsf{W}_{m-1} = \beta_m (\mathsf{Y}'_m)_{,i}, \tag{25}$$

and

$$\beta_m = \frac{1}{n + 2m - 2}. \tag{26}$$

Proof. Indeed, if we start with the decomposition (24) with $\mathsf{W}_{m\pm1} \in \mathcal{H}_{m\pm1}$ and apply the Laplacian to both sides we immediately obtain $(\mathsf{Y}'_m)_{,1} = \Delta\left(r^2 \mathsf{W}_{m-1}\right) = (n + 2m - 2)\,\mathsf{W}_{m-1}$, so that (25) follows. ☑

Notice, for future reference, that β_m can actually be considered as a meromorphic function of m.

We can iterate the result of the Proposition 3.1 to obtain the decomposition of $x_i x_j \mathsf{Y}'_m$ if $m \geq 2$. Indeed, if we use (24) then

$$\begin{aligned} x_i x_j \mathsf{Y}'_m &= x_j \mathsf{W}_{m+1} + r^2 x_j \mathsf{W}_{m-1} \\ &= (x_j \mathsf{W}_{m+1} - r^2 \beta_{m+1} (\mathsf{W}_{m+1})_{,i}) + r^2 \beta_{m+1} (\mathsf{W}_{m+1})_{,i} \\ &\quad + r^2 x_j \left(\mathsf{W}_{m-1} - r^2 \beta_{m-1} (\mathsf{W}_{m-1})_{,i} + r^2 \beta_{m-1} (\mathsf{W}_{m-1})_{,i} \right) . \end{aligned}$$

Employing (25) we therefore obtain the following decomposition.

Proposition 3.2. *If* $\mathsf{Y}'_m \in \mathcal{H}_m$, $m \geq 2$, *then the Gauss decomposition of* $x_i x_j \mathsf{Y}'_m$ *is*

$$x_i x_j \mathsf{Y}'_m = \mathsf{Z}_{m+2} + r^2 \mathsf{Z}_m + r^4 \mathsf{Z}_{m-2} \, , \tag{27}$$

where

$$\begin{aligned} \mathsf{Z}_{m+2} = x_i x_j \mathsf{Y}'_m \\ - \beta_{m+1} r^2 (\delta_{ij} \mathsf{Y}'_m + x_i (\mathsf{Y}'_m)_{,j} + x_j (\mathsf{Y}'_m)_{,i}) + \beta_{m+1} \beta_m r^4 (\mathsf{Y}'_m)_{,ij} , \end{aligned} \tag{28}$$

$$\mathsf{Z}_m = \beta_{m+1} (\delta_{ij} \mathsf{Y}'_m + x_i (\mathsf{Y}'_m)_{,j} + x_j (\mathsf{Y}'_m)_{,i}) - 2 \beta_{m+1} \beta_{m-1} r^2 (\mathsf{Y}'_m)_{,ij} , \tag{29}$$

and

$$\mathsf{Z}_{m-2} = \beta_m \beta_{m-1} (\mathsf{Y}'_m)_{,ij} . \tag{30}$$

Consider now the decomposition of the product $\mathsf{Y}_2 \mathsf{Y}'_m$. The elements of $\mathcal{H}_2$ have the form $\mathsf{Y}_2 = \sum_{i,j=1}^n a_{ij} x^i x^j$ with $\sum_{i=1}^n a_{ii} = 0$ and therefore the Proposition 3.2 yields the general formula that follows.

Proposition 3.3. *If* $\mathsf{Y}_2 \in \mathcal{H}_2$, $\mathsf{Y}'_m \in \mathcal{H}_m$, $m \geq 2$, *then the Gauss decomposition of* $\mathsf{Y}_2 \mathsf{Y}'_m$ *is*

$$\mathsf{Y}_2 \mathsf{Y}'_m = \mathsf{Z}_{m+2} + r^2 \mathsf{Z}_m + r^4 \mathsf{Z}_{m-2} \, , \tag{31}$$

where

$$\mathsf{Z}_{m+2} = \left(\mathsf{Y}_2 \left(\mathbf{x}^2 \right) - 2 \beta_{m+1} r^2 \mathsf{Y}_2 \left(\mathbf{x} \nabla \right) + \beta_{m+1} \beta_m r^4 \mathsf{Y}_2 \left(\nabla^2 \right) \right) \mathsf{Y}'_m \, , \tag{32}$$

$$\mathsf{Z}_m = \left(2 \beta_{m+1} \mathsf{Y}_2 \left(\mathbf{x} \nabla \right) - 2 \beta_{m+1} \beta_{m-1} r^2 \mathsf{Y}_2 \left(\nabla^2 \right) \right) \mathsf{Y}'_m \, , \tag{33}$$

and

$$\mathsf{Z}_{m-2} = \beta_m \beta_{m-1} \mathsf{Y}_2 \left(\nabla^2 \right) \mathsf{Y}'_m \, . \tag{34}$$

Observe that the formulas of this last proposition do not hold if we try to decompose the product $p \mathsf{Y}'_m$ with $p \in \mathcal{P}_2 \setminus \mathcal{H}_2$. An extremely simple example is provided by $p = r^2$. A more instructive example is provided by taking $p = x_i^2$, whose decomposition is given in the Proposition 3.2, but which we now show how to decompose in another way. Indeed, we can write

$$x_i^2 = \mathsf{Y}_2 + \frac{1}{n}r^2, \qquad \mathsf{Y}_2 = \left(x_i^2 - \frac{1}{n}r^2\right) \in \mathcal{H}_2 \,. \tag{35}$$

Hence $x_i^2\mathsf{Y}'_m = \mathsf{Y}_2\mathsf{Y}'_m - \frac{1}{n}r^2\mathsf{Y}'_m$, and the decomposition of $\mathsf{Y}_2\mathsf{Y}'_m$ follows from the Proposition 3.3 by observing that

$$\mathsf{Y}_2\left(\mathbf{x}\nabla\right)\mathsf{Y}'_m = x_i\nabla_i\mathsf{Y}'_m - \frac{m}{n}\mathsf{Y}'_m\,, \quad \mathsf{Y}_2\left(\nabla^2\right) = \nabla_i^2\mathsf{Y}'_m\,. \tag{36}$$

Naturally we obtain (28)-(30) with $i = j$ (not summed).

3.1. Zonal Harmonics

Let $\zeta \in \mathbb{S}$ be a fixed unit vector. There exists a unique monic polynomial of degree m, u_m, such that

$$\mathsf{U}_m^\zeta\left(\omega\right) = u_m\left(\omega \cdot \zeta\right) \in \mathcal{H}_m\left(\mathbb{S}\right). \tag{37}$$

The U_m^ζ and its multiples are the *zonal harmonics* of degree m, the spherical harmonics that are constant on circles perpendicular to ζ. The multiples of the polynomials u_m are called *ultraspherical polynomials,* and are actually multiples of the Chebychev polynomials if $n = 2$, the Legendre polynomials if $n = 3$, and in general of Gegenbauer polynomials for any m [3, 17, 16]. The corresponding solid harmonics are given as

$$\mathsf{U}_m^\zeta\left(\mathbf{x}\right) = \mathsf{U}_m^\zeta\left(r\omega\right) = r^m u_m\left(\omega\cdot\zeta\right) = r^m u_m\left(\mathbf{x}\cdot\zeta/r\right). \tag{38}$$

Since the polynomials u_m do not depend on ζ, we may take $\zeta = (1, 0, \ldots, 0)$, so that $\mathbf{x}\cdot\zeta = x_1$, and we will do so in the following, employing the simpler notation U_m.

We have $u_0(x) = 1$, $u_1(x) = x$, while we encountered u_2 in (35) since $\mathsf{U}_2 = \mathsf{Y}_2$, that is, $u_2(x) = x^2 - 1/n$. We may use the Proposition 3.1 to find a recursion relation for the u_m since U_{m+1} must be the part in the Gauss decomposition of $x_1\mathsf{U}_m$ that belongs to $\mathcal{H}_{m+1}$. Therefore,

$$\mathsf{U}_{m+1}\left(\mathbf{x}\right) = x_1\mathsf{U}_m - r^2\beta_m\nabla_1(\mathsf{U}_m), \tag{39}$$

and consequently,

$$u_{m+1}\left(x\right) = xu_m - \beta_m L_m\left(u_m\right), \tag{40}$$

where $L_m\left(u\right)\left(x_1\right)$ is given as

$$\nabla_1\left(r^m u\left(\frac{x_1}{r}\right)\right)\Big|_{r=1} = mx_1 u\left(x_1\right) + \left(1 - x_1^2\right)u'\left(x_1\right). \tag{41}$$

We should also have that $\nabla_1(\mathsf{U}_m)$ must be a multiple of U_{m-1}; actually a simple computation gives

$$u_m = \frac{1}{\beta_{m-1} m\,(n+m-3)} L_m\,(u_m)\ , \tag{42}$$

so that

$$u_{m+1}\,(x) = x u_m - m\,(n+m-3)\,\beta_m \beta_{m-1} u_{m-1}\,. \tag{43}$$

These recursion relations give the expression

$$u_m\,(x) = x^m - \binom{m}{2}\beta_{m-1}x^{m-2} + \binom{m}{4}3\beta_{m-1}\beta_{m-2}x^{m-4} - \binom{m}{6}3\cdot 5\beta_{m-1}\beta_{m-2}\beta_{m-2}x^{m-6} + \cdots\,. \tag{44}$$

A particularly important multiple of the zonal harmonic $\mathsf{U}_m^{\zeta}\,(\omega)$ is $\mathsf{Z}\,(\zeta,\omega)$, the reproducing kernel of the finite dimensional Hilbert space $\mathcal{H}_m\,(\mathbb{S})$ with the L^2 inner product [3, Thm. 5.38], and is given as $\mathsf{Z}\,(\zeta,\omega) = c u_m\,(\omega\cdot\zeta)$, where $c = (m!\beta_m\beta_{m-1}\cdots\beta_1)^{-1}$:

$$\mathsf{Z}\,(\zeta,\mathbf{x}) = \frac{1}{\beta_m}\sum_{q=0}^{[\![m/2]\!]} (-1)^q\,\frac{n\,(n+2)\cdots(n+2m-2q-4)}{2^q q!\,(m-2q)!}\,(\mathbf{x}\cdot\zeta)^{m-2q}\,|\mathbf{x}|^{2q}\,. \tag{45}$$

4. The general formula

We shall now give the general formula for the Gauss decomposition of the product $\mathsf{Y}_k\mathsf{Y}'_m$, $k\le m$, of two harmonic polynomials. We start by defining several sets and constants that will appear in our formulas.

The sets $\mathfrak{B}_q^j$, $1\le q\le j+1$, are subsets with j elements of $\{q\in\mathbb{Z} : |q|\le j-1\}$ defined as

$$\mathfrak{B}_1^j = \{j-1,\ldots,0\}\ , \qquad \mathfrak{B}_{j+1}^j = \{0,\ldots,-(j-1)\}\,, \tag{46}$$

while if $1<q<j+1$,

$$\mathfrak{B}_q^j = \{j-q+1,\ldots,j-2q+3\}\cup\{j-2q+1,\ldots,-q+1\}\,. \tag{47}$$

For example $\mathfrak{B}_1^2 = \{1,0\}, \mathfrak{B}_2^2 = \{1,-1\}$, and $\mathfrak{B}_3^2 = \{0,-1\}$ while $\mathfrak{B}_1^3 = \{2,1,0\}$, $\mathfrak{B}_2^3 = \{2,0,-1\}$, $\mathfrak{B}_3^3 = \{1,0,-2\}$, and $\mathfrak{B}_4^3 = \{0,-1,-2\}$. It would also be convenient to use the extreme cases $\mathfrak{B}_1^0 = \varnothing$, $\mathfrak{B}_1^1 = \mathfrak{B}_2^1 = \{0\}$.

The constants $B_{j,q}^{(t)}$ are the products

$$B_{j,q}^{(t)} = \prod_{l\in\mathfrak{B}_q^j}\beta_{l+t}\,. \tag{48}$$

For instance $B_{2,1}^{(t)} = \beta_{t+1}\beta_t$, $B_{2,2}^{(t)} = \beta_{t+1}\beta_{t-1}$, and $B_{2,3}^{(t)} = \beta_t\beta_{t-1}$. We also have $B_{3,1}^{(t)} = \beta_{t+2}\beta_{t+1}\beta_t$, $B_{3,2}^{(t)} = \beta_{t+2}\beta_t\beta_{t-1}$, $B_{3,3}^{(t)} = \beta_{t+1}\beta_t\beta_{t-2}$, and $B_{3,4}^{(t)} = \beta_t\beta_{t-1}\beta_{t-2}$. If $j=0$ or 1 we obtain $B_{0,1}^{(t)} = 1$, $B_{1,1}^{(t)} = B_{2,1}^{(t)} = \beta_t$.

Theorem 4.1. *If $\mathsf{Y}_k \in \mathcal{H}_k$, $\mathsf{Y}'_m \in \mathcal{H}_m$, $m \geq k$, then the Gauss decomposition of $\mathsf{Y}_k\mathsf{Y}'_m$ is*

$$\mathsf{Y}_k\mathsf{Y}'_m = \sum_{q=0}^{k} r^{2q}\mathsf{Z}_{m+k-2q}, \tag{49}$$

where the $\mathsf{Z}_{m+k-2q} \in \mathcal{H}_{m+k-2q}$ are given as

$$\begin{aligned}&\mathsf{Z}_{m+k-2q}\\ &= \binom{k}{q}\sum_{j=q}^{k}\binom{k-q}{j-q}B_{j,q+1}^{(m+k-j)}\left(-1\right)^{j-q} r^{2j-2q}\mathsf{Y}_k\left(\mathbf{x}^{k-j}\nabla^j\right)\mathsf{Y}'_m\,.\end{aligned} \tag{50}$$

We shall give the proof of the theorem in Section 6. Presently we give several special cases of the general formula. Indeed, if $k=3$ we obtain the Gauss decomposition of $\mathsf{Y}_3\mathsf{Y}'_m = \mathsf{Z}_{m+3} + r^2\mathsf{Z}_{m+1} + r^4\mathsf{Z}_{m-1} + r^6\mathsf{Z}_{m-3}$ as

$$\begin{aligned}\mathsf{Z}_{m+3} = &\left(\mathsf{Y}_3\left(\mathbf{x}^3\right) - 3\beta_{m+2}r^2\mathsf{Y}_3\left(\mathbf{x}^2\nabla\right)\right.\\ &\left.+3\beta_{m+2}\beta_{m+1}r^4\mathsf{Y}_3\left(\mathbf{x}\nabla^2\right) - \beta_{m+2}\beta_{m+1}\beta_m r^6\mathsf{Y}_3\left(\nabla^3\right)\right)\mathsf{Y}'_m\,,\end{aligned} \tag{51}$$

$$\begin{aligned}\mathsf{Z}_{m+1} = &\left(3\beta_{m+2}\mathsf{Y}_3\left(\mathbf{x}^2\nabla\right)\right.\\ &\left.-6\beta_{m+2}\beta_m r^2\mathsf{Y}_3\left(\mathbf{x}\nabla^2\right) + 3\beta_{m+2}\beta_m\beta_{m-1}r^4\mathsf{Y}_3\left(\nabla^3\right)\right)\mathsf{Y}'_m\,,\end{aligned} \tag{52}$$

$$\begin{aligned}\mathsf{Z}_{m-1} = &\left(3\beta_{m+1}\beta_m\mathsf{Y}_3\left(\mathbf{x}\nabla^2\right)\right.\\ &\left.-3\beta_{m+1}\beta_m\beta_{m-2}r^2\mathsf{Y}_3\left(\nabla^3\right)\right)\mathsf{Y}'_m\,,\end{aligned} \tag{53}$$

$$\mathsf{Z}_{m-3} = \beta_m\beta_{m-1}\beta_{m-2}\mathsf{Y}_3\left(\nabla^3\right)\mathsf{Y}'_m\,. \tag{54}$$

As another interesting particular case, let us employ the general formula (50) with Y_k replaced with the zonal harmonic U_m of (38). We obtain that if $m \geq k$ and $\mathsf{U}_k\mathsf{Y}'_m = \sum_{q=0}^{k} r^{2q}\mathsf{Z}_{m+k-2q}$ then calling α_j the coefficients in the expansion (44),

$$\begin{aligned}\mathsf{Z}_{m-k}\left(\mathbf{x}\right) &= B_{k,k+1}^{(m)}\mathsf{U}_k\left(\nabla\right)\mathsf{Y}'_m\left(\mathbf{x}\right)\\ &= B_{k,k+1}^{(m)}\left(\nabla_1^k + \alpha_1\nabla_1^{k-2}\Delta + \alpha_1\nabla_1^{k-2}\Delta + \cdots\right)\mathsf{Y}'_m\left(\mathbf{x}\right)\\ &= B_{k,k+1}^{(m)}\nabla_1^k\mathsf{Y}'_m\left(\mathbf{x}\right) = \beta_m\beta_{m-1}\cdots\beta_{m-k+1}\nabla_1^k\mathsf{Y}'_m\left(\mathbf{x}\right)\,.\end{aligned} \tag{55}$$

Similarly,

$$\mathsf{Z}_{m-k-2}(\mathbf{x}) = k(B_{k-1,k}^{(m+1)}\mathsf{U}_k\left(\mathbf{x}\nabla^{(k-1)}\right) - B_{k,k}^{(m)}r^2\mathsf{U}_k(\nabla))\mathsf{Y}'_m(\mathbf{x}),$$

and employing the Example 2.6,

$$\begin{aligned}\mathsf{U}_k\left(\mathbf{x}\nabla^{(k-1)}\right)\mathsf{Y}'_m(\mathbf{x}) &= x_1\nabla_1^{k-1}\mathsf{Y}'_m(\mathbf{x}) - \frac{2\alpha_1}{k}D\left(\nabla_1^{k-2}\mathsf{Y}'_m(\mathbf{x})\right)\\ &= x_1\nabla_1^{k-1}\mathsf{Y}'_m(\mathbf{x}) - \frac{2\alpha_1}{k}(m-k+2)\nabla_1^{k-2}\mathsf{Y}'_m(\mathbf{x}),\end{aligned}$$

so that

$$\begin{aligned}\mathsf{Z}_{m-k-2}(\mathbf{x}) &= kB_{k-1,k}^{(m+1)}x_1\nabla_1^{k-1}\mathsf{Y}'_m(\mathbf{x}) \\ &\quad - k(k-1)B_{k-1,k}^{(m+1)}(m-k+2)\nabla_1^{k-2}\mathsf{Y}'_m(\mathbf{x}) - kB_{k,k}^{(m)}r^2\nabla_1^k\mathsf{Y}'_m(\mathbf{x}).\end{aligned} \tag{56}$$

5. An identity for the β_m

The fact that (32)-(34) is the Gauss decomposition of $\mathsf{Y}_2\mathsf{Y}'_m$ implies that we should have

$$\beta_{m+1}\beta_m - 2\beta_{m+1}\beta_{m-1} + \beta_m\beta_{m-1} = 0. \tag{57}$$

Similarly, (51)-(54) gives

$$-\beta_{m+2}\beta_{m+1}\beta_m + 3\beta_{m+2}\beta_m\beta_{m-1} - 3\beta_{m+1}\beta_m\beta_{m-2} + \beta_m\beta_{m-1}\beta_{m-2} = 0. \tag{58}$$

Both identities are easy to verify directly. Notice also that they will actually hold for all $m \in \mathbb{C}$, since they hold for all large enough integers and the left side is a rational function of m.

If we employ the general formula given in the Theorem 4.1 we therefore obtain the ensuing identity.

Proposition 5.1. *For all $m \in \mathbb{C}$,*

$$\sum_{q=0}^{k}\binom{k}{q}(-1)^{k-q}B_{k,q+1}^{(m)} = 0. \tag{59}$$

The case $k = 4$, that follows, should help to clarify the notation:

$$\begin{aligned}&\beta_{m+3}\beta_{m+2}\beta_{m+1}\beta_m - 4\beta_{m+3}\beta_{m+1}\beta_m\beta_{m-1} + 6\beta_{m+2}\beta_{m+1}\beta_{m-1}\beta_{m-2}\\ &- 4\beta_{m+1}\beta_m\beta_{m-1}\beta_{m-3} + \beta_m\beta_{m-1}\beta_{m-2}\beta_{m-3} = 0.\end{aligned}$$

6. Proof of the Theorem

In this section we give a proof of the Theorem 4.1. We shall proceed by induction on k. The case $k = 1$ has already being given in the Proposition 3.1. On the other hand, it is enough to show that if we assume the formula (50) for all $\mathsf{Y}_k \in \mathcal{H}_k$ then the same formula holds for *just one* harmonic polynomial $\mathsf{Y}_{k+1} \in \mathcal{H}_{k+1}$. Indeed, we may invoke the Funk-Hecke formula[4] as presented in [6], since it is clear that if $\mathsf{Y}_k\mathsf{Y}'_m = \sum_{q=0}^{k} r^{2q}\mathsf{Z}_{m+k-2q}$ then

$$\mathsf{Z}_{m+k-2q} = \sum_{j=q}^{k} r^{2j-2q} p_{k,j,q} \left(\mathbf{x}^{k-j} \vee \nabla^{j}\right) \mathsf{Y}'_m \, , \tag{60}$$

for some polynomials $p_{k,j,q} = \mathcal{P}_{k,j,q}\left(\mathsf{Y}_k\right)$ and the operators $\mathcal{P}_{k,j,q}$ are invariant under the group $SO\left(n\right)$. Hence

$$p_{k,j,q} = \lambda_{k,j,q} \mathsf{Y}_k \, , \tag{61}$$

for some constants, the *same* for all $\mathsf{Y}_k \in \mathcal{H}_k$. What this means is that if we are able to establish the formula for one $\mathsf{Y}_{k+1} \in \mathcal{H}_{k+1}$ then it will hold for *all* homogeneous harmonic polynomials of degree $k+1$.

We shall take $\mathsf{Y}_{k+1}\left(\mathbf{x}\right) = x_1 \mathsf{Y}_k\left(\widetilde{\mathbf{x}}\right)$ where $\widetilde{\mathbf{x}} = \left(0, x_2, \ldots, x_n\right)$, that is, where Y_k does not depend on x_1. This is possible as long as $n \geq 3$; the proof when $n = 2$ is simpler and will not be presented here. Let $m \geq k+1$ and $\mathsf{Y}'_m \in \mathcal{H}_m$. The induction hypothesis tells us that the Gauss decomposition of $\mathsf{Y}_k\mathsf{Y}'_m$ is given in (49)-(50). Let us write

$$\mathsf{Y}_{k+1}\mathsf{Y}'_m = \sum_{q=0}^{k+1} r^{2q} \mathsf{W}_{m+k+1-2q} \, . \tag{62}$$

We need to show that the W_{m+k-2q} are given by the formula corresponding to (50) applied to Y_{k+1} instead of Y_k. Employing the Proposition 3.1 and (49) we obtain

$$\mathsf{W}_{m+k+1} = x_1 \mathsf{Z}_{m+k} - \beta_{m+k} r^2 \nabla_1 \mathsf{Z}_{m+k} \, , \qquad \mathsf{W}_{m-k-1} = \beta_{m-k} \nabla_1 \mathsf{Z}_{m-k} \, , \tag{63}$$

while for $1 \leq q \leq k$, $\mathsf{W}_{m+k+1-2q}$ equals

$$x_1 \mathsf{Z}_{m+k-2q} - \beta_{m+k-2q} r^2 \nabla_1 \mathsf{Z}_{m+k-2q} + \beta_{m+k-2q+2} \nabla_1 \mathsf{Z}_{m+k-2q+2} \, . \tag{64}$$

We immediately obtain the required formula for W_{m-k-1}, namely:

$$\beta_{m-k} \nabla_1 (B^{(m)}_{k,k+1} \mathsf{Y}_k \left(\nabla^k\right) \mathsf{Y}'_m) = B^{(m)}_{k+1,k+2} \mathsf{Y}_{k+1} \left(\nabla^{k+1}\right) \mathsf{Y}'_m \, . \tag{65}$$

[4]This celebrated formula was first given for integral transforms in three variables in [11, 12], and for any number of variables in [4].

The expression for W_{m+k+1} is obtained as follows, where we use (49) and the fact that $\nabla_1 \mathsf{Y}_k = 0$:

$$\begin{aligned}\mathsf{W}_{m+k+1} &= x_1 \sum_{j=0}^{k} \binom{k}{j} B_{j,1}^{(m+k-j)} (-1)^j r^{2j} \mathsf{Y}_k \left(\mathbf{x}^{k-j} \nabla^j\right) \mathsf{Y}'_m \\ &- \beta_{m+k} r^2 \sum_{j=0}^{k} \binom{k}{j} B_{j,1}^{(m+k-j)} (-1)^j \left(2j x_1 r^{2j-2} \mathsf{Y}_k \left(\mathbf{x}^{k-j} \nabla^j\right)\right. \\ &\left. + r^{2j} \mathsf{Y}_k \left(\mathbf{x}^{k-j} \nabla^j\right) \nabla_1\right) \mathsf{Y}'_m ,\end{aligned}$$

so that $\mathsf{W}_{m+k+1} = \sum_{j=0}^{k+1} r^{2j} p_j (\mathbf{x}, \nabla) \mathsf{Y}'_m$. Notice that

$$p_0 = B_{0,1}^{(m+k)} x_1 \mathsf{Y}_k \left(\mathbf{x}^{k-j} \nabla^j\right) = B_{0,1}^{(m+k+1)} \mathsf{Y}_{k+1} \left(\nabla^{k+1}\right), \tag{66}$$

since $B_{0,1}^{(t)} = 1$ for all t. Also p_{k+1} is given as

$$-\beta_{m+k} B_{k,1}^{(m+k)} (-1)^k \mathsf{Y}_k \left(\nabla^k\right) \nabla_1 = B_{k+1,1}^{(m+k+1)} (-1)^{k+1} \mathsf{Y}_{k+1} \left(\nabla^{k+1}\right). \tag{67}$$

When $1 \leq j \leq k$ we obtain

$$p_j (\mathbf{x}, \nabla) = A_j x_1 \mathsf{Y}_k \left(\mathbf{x}^{k-j} \nabla^j\right) + B_j \mathsf{Y}_k \left(\mathbf{x}^{k+1-j} \nabla^{j-1}\right) \nabla_1, \tag{68}$$

where

$$\begin{aligned}A_j &= \binom{k}{j} B_{j,1}^{(m+k-j)} (-1)^j - \binom{k}{j} 2j \beta_{m+k} B_{j,1}^{(m+k-j)} (-1)^j \\ &= \binom{k}{j} B_{j,1}^{(m+k+1-j)} (-1)^j ,\end{aligned}$$

since $1 - 2j\beta_{m+k} = \beta_{m+k}/\beta_{m+k-j}$, and where

$$B_j = -\binom{k}{j-1} \beta_{m+k} B_{j-1,1}^{(m+k-j+1)} (-1)^j = \binom{k}{j-1} B_{j,1}^{(m+k+1-j)} (-1)^j .$$

But we observe that

$$\mathsf{Y}_{k+1} \left(\mathbf{x}^{k+1-j} \nabla^j\right) = \frac{k+1-j}{k+1} x_1 \mathsf{Y}_k \left(\mathbf{x}^{k-j} \nabla^j\right) + \frac{j}{k+1} \mathsf{Y}_k \left(\mathbf{x}^{k+1-j} \nabla^{j-1}\right) \nabla_1 , \tag{69}$$

so that,

$$p_j (\mathbf{x}, \nabla) = \binom{k+1}{j} B_{j,1}^{(m+(k+1)-j)} (-1)^j \mathsf{Y}_{k+1} \left(\mathbf{x}^{k+1-j} \nabla^j\right). \tag{70}$$

Therefore W_{m+k+1} has the required form.

The proof of the formula for $\mathsf{W}_{m+k+1-2q}$ for $1 \leq q \leq k$ is very similar and consequently will be omitted.

7. The integral of the product of three spherical harmonics

In this section we shall apply the formulas for the product of harmonic polynomials to obtain identities for integrals of the type

$$\int_{\mathbb{S}} \mathsf{Y}_k(\omega)\, \mathsf{Y}'_m(\omega)\, \mathsf{Y}''_l(\omega)\, \mathrm{d}\sigma(\omega)\ , \tag{71}$$

involving the product of three spherical harmonics $\mathsf{Y}_k \in \mathcal{H}_k$, $\mathsf{Y}'_m \in \mathcal{H}_m$, $\mathsf{Y}''_l \in \mathcal{H}_l$. Henceforth we will assume that $m \geq k$ and $m \geq l$; naturally the formula (49) shows that with this restriction we must have $l = m + k - 2q$ for some q that satisfies $k \geq q \geq k/2$ if the integral is not zero, and in that case the integral $\int_{\mathbb{S}} \mathsf{Y}_k(\omega)\, \mathsf{Y}'_m(\omega)\, \mathsf{Y}''_{m+k-2q}(\omega)\, \mathrm{d}\sigma(\omega)$ equals

$$\int_{\mathbb{S}} \mathsf{Z}_{m+k-2q}(\omega)\, \mathsf{Y}''_{m+k-2q}(\omega)\, \mathrm{d}\sigma(\omega)\,. \tag{72}$$

Let us start with the case $q = k = m$, $\mathsf{Y}''_{m+k-2q} = 1$, the case of the product of two spherical harmonics of the same order. We obtain

$$\begin{aligned} \int_{\mathbb{S}} \mathsf{Y}_m(\omega)\, \mathsf{Y}'_m(\omega)\, \mathrm{d}\sigma(\omega) &= \int_{\mathbb{S}} \mathsf{Z}_{m+k-2q}(\omega)\, \mathrm{d}\sigma(\omega) \\ &= C B^{(m)}_{m,m+1} \mathsf{Y}_m(\nabla)\, \mathsf{Y}'_m(\mathbf{x}) \\ &= C \beta_m \cdots \beta_1 \left\{ \mathsf{Y}_m, \overline{\mathsf{Y}}'_m \right\}, \end{aligned} \tag{73}$$

where the inner product $\left\{ \mathsf{Y}_m, \overline{\mathsf{Y}}'_m \right\}$ is the constant function $\mathsf{Y}_m(\nabla)\, \mathsf{Y}'_m(\mathbf{x})$ [10] and C is the area of $\mathbb{S}$, recovering [5, Prop. 3.3].

We can now give a general identity for the integral (71).

Proposition 7.1. *Let* $\mathsf{Y}_k \in \mathcal{H}_k$, $\mathsf{Y}'_m \in \mathcal{H}_m$, $\mathsf{Y}''_l \in \mathcal{H}_l$, $m \geq k$, $m \geq l$. *The integral vanishes unless* $l = m + k - 2q$, $k \geq q \geq k/2$, *and then*

$$\int_{\mathbb{S}} \mathsf{Y}_k(\omega)\, \mathsf{Y}'_m(\omega)\, \mathsf{Y}''_{m+k-2q}(\omega)\, \mathrm{d}\sigma(\omega) = \binom{k}{q} B^{(m+k-q)}_{k,k+1} \int_{\mathbb{S}} \mathsf{X}_{m-q}(\omega)\, \mathrm{d}\sigma(\omega)\,, \tag{74}$$

where

$$\mathsf{X}_{m-q}(\omega) = \mathsf{Y}_k\left(\nabla^q_{\mathbf{x}} \nabla^{k-q}_{\mathbf{y}}\right) \left(\mathsf{Y}'_m(\mathbf{x})\, \mathsf{Y}''_{m+k-2q}(\mathbf{y})\right)\Big|_{\mathbf{x}=\mathbf{y}=\omega}\,. \tag{75}$$

Proof. Indeed, the integral in the left of (74) equals (72), so that (50) gives it as

$$\binom{k}{q} B^{(m+k-q)}_{q,q+1} \int_{\mathbb{S}} \mathsf{Y}_k\left(\mathbf{x}^{k-q} \nabla^q\right) \mathsf{Y}'_m(\mathbf{x})\Big|_{\mathbf{x}=\omega} \mathsf{Y}''_{m+k-2q}(\omega)\, \mathrm{d}\sigma(\omega)\,, \tag{76}$$

and employing (73), we obtain

$$M \left\{ \mathsf{Y}_k\left(\mathbf{x}^{k-q} \nabla^q\right) \mathsf{Y}'_m(\mathbf{x})\,, \mathsf{Y}''_{m+k-2q}(\mathbf{x}) \right\}, \tag{77}$$

where $M=\binom{k}{q}CB_{q,q+1}^{(m+k-q)}B_{m+k-2q,m+k-2q+1}^{(m+k-2q)}$. But [10] $\{x_ip(\mathbf{x}),q(\mathbf{x})\}=\{p(\mathbf{x}),\nabla_iq(\mathbf{x})\}$, so that the integral becomes

$$M\left\{\mathsf{Y}_k\left(\nabla_{\mathbf{x}}^q\nabla_{\mathbf{y}}^{k-q}\right)\left(\mathsf{Y}_m'(\mathbf{x}),\mathsf{Y}_{m+k-2q}''(\mathbf{y})\right)\big|_{\mathbf{x}=\mathbf{y}}\right\}, \tag{78}$$

and thus using (73) again we arrive at (74), since

$$\frac{M}{CB_{m-q,m-q+1}^{(m-q)}}=\frac{B_{q,q+1}^{(m+k-q)}B_{m+k-2q,m+k-2q+1}^{(m+k-2q)}}{B_{m-q,m-q+1}^{(m-q)}}=B_{k,k+1}^{(m+k-q)}. \tag{79}$$

☑

We now consider several illustrations of these formulas. If $k=2$ and $\mathsf{Y}_2=x_ix_j$, $i\neq j$, we obtain for $q=1$,

$$\int_{\mathbb{S}}\omega_i\omega_j\mathsf{Y}_m'\mathsf{Y}_m''\,\mathrm{d}\sigma=\beta_{m+1}\beta_m\int_{\mathbb{S}}(\nabla_i\mathsf{Y}_m'\nabla_j\mathsf{Y}_m''+\nabla_j\mathsf{Y}_m'\nabla_i\mathsf{Y}_m'')\,\mathrm{d}\sigma, \tag{80}$$

and for $q=2$,

$$\int_{\mathbb{S}}\omega_i\omega_j\mathsf{Y}_m'\mathsf{Y}_{m-2}''\,\mathrm{d}\sigma=\beta_m\beta_{m-1}\int_{\mathbb{S}}(\nabla_i\nabla_j\mathsf{Y}_m')\mathsf{Y}_{m-2}''\,\mathrm{d}\sigma. \tag{81}$$

On the other hand, the integral $\int_{\mathbb{S}}\omega_i^2\mathsf{Y}_m'\mathsf{Y}_m''\,\mathrm{d}\sigma$ can be handled if we use (35); the result is

$$\beta_{m+1}\beta_m\int_{\mathbb{S}}(\nabla_i\mathsf{Y}_m'\nabla_i\mathsf{Y}_m''-\frac{1}{n}\sum_{l=1}^{n}\nabla_j\mathsf{Y}_m'\nabla_j\mathsf{Y}_m''+\frac{1}{n}\mathsf{Y}_m'\mathsf{Y}_m'')\,\mathrm{d}\sigma, \tag{82}$$

which *does not follow* if we put $i=j$ in (80). Curiously the result of putting $i=j$ in (81) yields the correct result

$$\int_{\mathbb{S}}\omega_i^2\mathsf{Y}_m'\mathsf{Y}_{m-2}''\,\mathrm{d}\sigma=\beta_m\beta_{m-1}\int_{\mathbb{S}}(\nabla_i^2\mathsf{Y}_m')\mathsf{Y}_{m-2}''\,\mathrm{d}\sigma. \tag{83}$$

When $k=3$ and $\mathsf{Y}_2=x_ix_jx_l$, $i\neq j\neq l\neq i$, we obtain that $\int_{\mathbb{S}}\omega_i\omega_j\omega_l\mathsf{Y}_m'\mathsf{Y}_{m-1}''\,\mathrm{d}\sigma$ equals

$$\beta_{m+1}\beta_m\beta_{m-1}\int_{\mathbb{S}}(\nabla_i\nabla_j\mathsf{Y}_m'\nabla_l\mathsf{Y}_{m-1}''+\nabla_i\nabla_l\mathsf{Y}_m'\nabla_j\mathsf{Y}_{m-1}''+\nabla_j\nabla_l\mathsf{Y}_m'\nabla_i\mathsf{Y}_{m-1}'')\,\mathrm{d}\sigma, \tag{84}$$

while

$$\int_{\mathbb{S}}\omega_i\omega_j\omega_l\mathsf{Y}_m'\mathsf{Y}_{m-3}''\,\mathrm{d}\sigma=\beta_m\beta_{m-1}\beta_{m-2}\int_{\mathbb{S}}(\nabla_i\nabla_j\nabla_l\mathsf{Y}_m')\mathsf{Y}_{m-3}''\,\mathrm{d}\sigma. \tag{85}$$

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Department of Mathematics
Louisiana State University
Baton Rouge, LA
70803 USA
e-mail: restrada@math.lsu.edu