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Pillai's problem with Padovan numbers and powers of two

El problema de Pillai con números de Padovan y potencias de dos

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Abstract. Let $(P_n)_{n\geqslant 0}$ be the *Padovan sequence* given by $P_0 = 0$, $P_1 = P_2 = 1$ and the recurrence formula $P_{n+3} = P_{n+1} + P_n$ for all $n \geqslant 0$. In this note we study and completely solve the Diophantine equation

$$P_n - 2^m = P_{n_1} - 2^{m_1}$$

in non-negative integers (n, m, n_1, m_1).

Key words and phrases. Padovan sequence, Pillai's problem, linear forms in logarithms, reduction method.

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Resumen. Sea $(P_n)_{n\geqslant 0}$ la *sucesión de Padovan* dada mediante $P_0 = 0$, $P_1 = P_2 = 1$ y la fórmula de recurrencia $P_{n+3} = P_{n+1} + P_n$ para todo $n \geqslant 0$. En esta nota estudiamos y resolvemos completamente la ecuación diofántica

$$P_n - 2^m = P_{n_1} - 2^{m_1}$$

en enteros no negativos (n, m, n_1, m_1).

Palabras y frases clave. Sucesión de Padovan, Problema de Pillai, Formas lineales en logaritmos, método de reducción.

1. Introduction

Let a, b be two fixed positive integers and consider the Diophantine equation

$$a^n - b^m = a^{n_1} - b^{m_1} \tag{1}$$

in positive integers (n, m, n_1, m_1) with $(n, m) \neq (n_1, m_1)$. In particular, look for the integers which can be written as the difference of a power of a and a power of b in at least two distinct ways. In [11], Herschfeld proved that in the case $(a, b) = (2, 3)$ equation (1) has finitely many solutions. In [13], Pillai extended Herschfeld's result to the case $a, b \geqslant 2$ being coprime integers. In both cases the result is ineffective. In [14], Pillai conjectures that in the case $(a, b) = (2, 3)$ all solutions of equation (1) are $(3, 2, 1, 1)$, $(5, 3, 3, 1)$ and $(8, 5, 4, 1)$. This conjecture remained open for about 37 years and it was confirmed by Stroeker and Tijdeman in [17] by using Baker's theory on linear forms in logarithms of algebraic numbers.

Recently, the above problem now known as the *Pillai problem* was posed in the context of linear recurrence sequences. That is, let $\mathbf{U} := (U_n)_{n\geqslant 0}$ and $\mathbf{V} := (V_m)_{m\geqslant 0}$ be two linearly recurrence sequences of integers and look at the diophantine equation

$$U_n - V_m = U_{n_1} - V_{m_1} \tag{2}$$

in positive integers (n, m, n_1, m_1) with $(n, m) \neq (n_1, m_1)$. In particular, find the integers which can be written as a difference of an element of $\mathbf{U}$ and an element of $\mathbf{V}$ in at least two distinct ways. This version was started in [8] by Ddamulira, Luca and Rakotomalala where they take $\mathbf{U}$ being the Fibonacci sequence and $\mathbf{V}$ being the sequence of powers of 2. Many other cases have been studied, see for example [5, 3, 7, 10]. In [6] it is proved that, under some natural conditions on $\mathbf{U}$ and $\mathbf{V}$ equation (2) has finitely many solutions and all of them are effectively computable.

Now, let $(P_n)_{n\geqslant 0}$ be the *Padovan sequence*, named after the architect R. Padovan. It is a ternary recurrence sequence given by $P_0 = 0$, $P_1 = P_2 = 1$ and the recurrence formula

$$P_{n+3} = P_{n+1} + P_n, \qquad \text{for all} \quad n \geqslant 0.$$

This is A000931 sequence in [16]. Its first few terms are

$$0,\ 1,\ 1,\ 1,\ 2,\ 2,\ 3,\ 4,\ 5,\ 7,\ 9,\ 12,\ 16,\ 21,\ 28,\ 37,\ 49,\ 65,\ 86,\ 114,\ 151,\ \ldots$$

In this note, we study another particular case of equation (2) namely with Padovan numbers and powers of 2. More precisely, we solve the equation

$$P_n - 2^m = P_{n_1} - 2^{m_1} \tag{3}$$

in non-negative integers (n, m, n_1, m_1) with $(n, m) \neq (n_1, m_1)$. Since $P_1 = P_2 = P_3 = 1$ we assume that $n \neq 1, 2$ and $n_1 \neq 1, 2$. That is, whenever we think of 1 as a member of the Padovan sequence we think of it as being P_3. In the same way, $n \neq 4$ and $n_1 \neq 4$. Then, with these conventions, our result is the following:

Theorem 1.1. *All non-negative integer solutions* (n, m, n_1, m_1) *of equation* (3) *belong to the set*

$$\left\{\begin{array}{lllll}
(3,1,0,0), & (5,1,3,0), & (5,2,0,1), & (6,1,5,0), & (6,2,0,0),\\
(6,2,3,1), & (7,1,6,0), & (7,2,3,0), & (7,2,5,1), & (7,3,0,2),\\
(8,1,7,0), & (8,2,5,0), & (8,2,6,1), & (8,3,3,2), & (9,2,7,0),\\
(9,2,8,1), & (9,3,0,0), & (9,3,3,1), & (9,3,6,2), & (10,2,9,1),\\
(10,3,5,0), & (10,3,6,1), & (10,3,8,2), & (10,4,3,3), & (11,2,10,0),\\
(11,3,8,0), & (11,4,0,2), & (11,4,7,3), & (12,3,10,0), & (12,3,11,2),\\
(12,4,3,0), & (12,4,5,1), & (12,4,7,2), & (12,5,0,4), & (13,4,9,1),\\
(13,4,10,2), & (13,5,8,4), & (14,3,13,0), & (14,4,12,2), & (14,5,0,2),\\
(14,5,7,3), & (14,5,11,4), & (15,5,9,1), & (15,5,10,2), & (15,5,13,4),\\
(15,6,8,5), & (16,4,15,2), & (16,5,13,2), & (16,6,3,4), & (17,5,15,2),\\
(17,5,16,4), & (17,6,5,0), & (17,6,6,1), & (17,6,8,2), & (17,6,10,3),\\
(17,7,3,6), & (19,5,18,2), & (19,7,5,4), & (22,9,10,8), & (27,10,17,3),\\
(30,12,24,11) & & & &
\end{array}\right\}.$$

The set of integers which can be written as the difference of a Padovan number and a power of 2 *in at least two distinct ways is*

$$\{-1583, -247, -63, -27, -16, -15, -14, -11, -7, -4, -3, -2, -1,$$
$$0, 1, 2, 3, 4, 5, 8, 12, 17, 20, 33, 57, 82\}.$$

The representations of these numbers are

$$\begin{aligned}
-1583 &= P_{30} - 2^{12} = P_{24} - 2^{11};\\
-247 &= P_{22} - 2^{9} = P_{10} - 2^{8};\\
-63 &= P_{17} - 2^{7} = P_{3} - 2^{6};\\
-27 &= P_{15} - 2^{6} = P_{8} - 2^{5};\\
-16 &= P_{12} - 2^{5} = P_{0} - 2^{4};\\
-15 &= P_{16} - 2^{6} = P_{3} - 2^{4};\\
-14 &= P_{19} - 2^{7} = P_{5} - 2^{4};\\
-11 &= P_{13} - 2^{5} = P_{8} - 2^{4};\\
-7 &= P_{10} - 2^{4} = P_{3} - 2^{3};\\
-4 &= P_{14} - 2^{5} = P_{11} - 2^{4} = P_{7} - 2^{3} = P_{0} - 2^{2};\\
-3 &= P_{8} - 2^{3} = P_{3} - 2^{2};\\
-2 &= P_{5} - 2^{2} = P_{0} - 2^{1};\\
-1 &= P_{9} - 2^{3} = P_{6} - 2^{2} = P_{3} - 2^{1} = P_{0} - 2^{0};
\end{aligned}$$

$$
\begin{aligned}
0 &= P_{12} - 2^4 = P_7 - 2^2 = P_5 - 2^1 = P_3 - 2^0;\\
1 &= P_{17} - 2^6 = P_{10} - 2^3 = P_8 - 2^2 = P_6 - 2^1 = P_5 - 2^0;\\
2 &= P_7 - 2^1 = P_6 - 2^0;\\
3 &= P_9 - 2^2 = P_8 - 2^1 = P_7 - 2^0;\\
4 &= P_{11} - 2^3 = P_8 - 2^0;\\
5 &= P_{15} - 2^5 = P_{13} - 2^4 = P_{10} - 2^2 = P_9 - 2^1;\\
8 &= P_{12} - 2^3 = P_{11} - 2^2 = P_{10} - 2^0;\\
12 &= P_{14} - 2^4 = P_{12} - 2^2;\\
17 &= P_{16} - 2^5 = P_{13} - 2^2;\\
20 &= P_{14} - 2^3 = P_{13} - 2^0;\\
33 &= P_{17} - 2^5 = P_{16} - 2^4 = P_{15} - 2^2;\\
57 &= P_{27} - 2^{10} = P_{17} - 2^3;\\
82 &= P_{19} - 2^5 = P_{18} - 2^2.
\end{aligned}
$$

2. Tools

In this section, we gather the tools we need to prove Theorem 1.1. The first one is a lower bound for linear forms in logarithms due to Matveev. Let α be an algebraic number of degree d, let $a > 0$ be the leading coefficient of its minimal polynomial over $\mathbb{Z}$ and let $\alpha = \alpha^{(1)}, \dots, \alpha^{(d)}$ denote its conjugates. The *logarithmic height* of α is defined as

$$h(\alpha) = \frac{1}{d}\left(\log a + \sum_{i=1}^{d} \log\max\left\{|\alpha^{(i)}|, 1\right\}\right).$$

This height satisfies the following basic properties. For α, β algebraic numbers and $m \in \mathbb{Z}$ we have

- $h(\alpha + \beta) \leqslant h(\alpha) + h(\beta) + \log(2)$;
- $h(\alpha\beta) \leqslant h(\alpha) + h(\beta)$;
- $h(\alpha^m) = |m|h(\alpha)$.

Now let $\mathbb{L}$ be a real number field of degree $d_{\mathbb{L}}$, $\alpha_1, \dots, \alpha_\ell \in \mathbb{L}$ and $b_1, \dots, b_\ell \in \mathbb{Z} \setminus \{0\}$. Let $B \geqslant \max\{|b_1|, \dots, |b_\ell|\}$ and

$$\Lambda = \alpha_1^{b_1} \cdots \alpha_\ell^{b_\ell} - 1.$$

Let $A_1, \ldots, A_\ell$ be real numbers with

$$A_i \geqslant \max\{d_{\mathbb{L}}\, h(\alpha_i), |\log \alpha_i|, 0.16\}, \qquad i = 1, 2, \ldots, \ell.$$

The following result is due to Matveev in [12] (see also Theorem 9.4 in [4]).

Theorem 2.1. *Assume that $\Lambda \neq 0$. Then*

$$\log|\Lambda| > -1.4 \cdot 30^{\ell+3} \cdot \ell^{4.5} \cdot d_{\mathbb{L}}^2 \cdot (1 + \log d_{\mathbb{L}}) \cdot (1 + \log B) A_1 \cdots A_\ell.$$

In this note we always use $\ell = 3$. Further, $\mathbb{L} = \mathbb{Q}(\gamma)$ has degree $d_{\mathbb{L}} = 3$, where γ is defined at the beginning of Section 3. Thus, once and for all we fix the constant

$$C := 1.4 \cdot 30^{3+3} \cdot 3^{4.5} \cdot 3^2 \cdot (1 + \log 3) \approx 2.70443 \times 10^{12}.$$

Our second tool is a version of the reduction method of Baker-Davenport based on Lemma in [1]. We shall use the one given by Bravo, Gómez and Luca in [2]. For a real number x, we write $\|x\|$ for the distance from x to the nearest integer.

Lemma 2.2. *Let M be a positive integer. Let $\tau,\ \mu,\ A > 0,\ B > 1$ be given real numbers. Assume that p/q is a convergent of τ such that $q > 6M$ and $\varepsilon := \|\mu q\| - M\|\tau q\| > 0$. Then there is no solution to the inequality*

$$0 < |n\tau - m + \mu| < \frac{A}{B^w}$$

in positive integers n, m and w satisfying

$$n \leqslant M \quad \text{and} \quad w \geqslant \frac{\log(Aq/\varepsilon)}{\log(B)}.$$

Lemma 2.2 is a slight variation of the one given by Dujella and Pethő in [9]. Finally, the following result will be very useful. This is Lemma 7 in [15].

Lemma 2.3. *If $m \geqslant 1$, $T > (4m^2)^m$ and $T > x/(\log x)^m$. Then*

$$x < 2^m T (\log T)^m.$$

3. Proof of Theorem 1.1

To start with, we recall some properties of the Padovan sequence. For a complex number z we write $\overline{z}$ for its complex conjugate. Let $\omega \neq 1$ be a cubic root of 1. Put

$$\gamma := \sqrt[3]{\frac{9+\sqrt{69}}{18}} + \sqrt[3]{\frac{9-\sqrt{69}}{18}}, \quad \delta := \omega\sqrt[3]{\frac{9+\sqrt{69}}{18}} + \overline{\omega}\sqrt[3]{\frac{9-\sqrt{69}}{18}}.$$

It is clear that $\gamma, \delta, \overline{\delta}$ are the roots of the $\mathbb{Q}$-irreducible polynomial $X^3 - X - 1$. We also have the Binet formula

$$P_n = c_1\gamma^n + c_2\delta^n + c_3\overline{\delta}^n, \tag{4}$$

which holds for all $n \geqslant 0$, where

$$c_1 = \frac{\gamma(\gamma+1)}{2\gamma+3}, \qquad c_2 = \frac{\delta(\delta+1)}{2\delta+3}, \qquad c_3 = \overline{c_2}. \tag{5}$$

Formula (4) follows from the general theorem on linear recurrence sequences since the above polynomial is the characteristic polynomial of the Padovan sequence. We note that

$$\gamma = 1.32471\ldots, \; |\delta| = 0.86883\ldots, \; c_1 = 0.54511\ldots, \; |c_2| = 0.28241\ldots$$

Further, the inequalities

$$\gamma^{n-3} \leqslant P_n \leqslant \gamma^{n-1}, \tag{6}$$

hold for all $n \geqslant 1$. These can be proved by induction.

We start with the study of equation (3) in non-negative integers (n, m, n_1, m_1) with $(n, m) \neq (n_1, m_1)$ where, as we have said, $n \neq 1, 2, 4$ and $n_1 \neq 1, 2, 4$. We note, if $m = m_1$ then $P_n = P_{n_1}$ which implies $n = n_1$, a contradiction. Thus we assume $m > m_1$. Rewriting equation (3) as

$$P_n - P_{n_1} = 2^m - 2^{m_1} \tag{7}$$

we observe the right-hand is positive. So, the left-hand side is also positive and therefore, $n > n_1$. Now, we compare both sides of (7) using (6). We have

$$\gamma^{n-8} \leqslant P_n - P_{n_1} = 2^m - 2^{m_1} < 2^m.$$

Indeed, the left-hand side inequality is clear if $n_1 = 0$. If $n_1 = 3$, $n \geqslant 5$. For $n = 5$ it is also clear and for $n \geqslant 6$ we have $P_n - P_{n_1} \geqslant P_n - P_{n-1} = P_{n-5} \geqslant \gamma^{n-8}$. The right-hand side inequality is clear. Thus $\gamma^{n-8} < 2^m$. In a similar way,

$$\gamma^{n-1} \geqslant P_n \geqslant P_n - P_{n_1} = 2^m - 2^{m_1} > 2^{m-1}.$$

Thus,

$$(n-8)\,\frac{\log\gamma}{\log 2} \leqslant m \qquad \text{and} \qquad (n-1)\,\frac{\log\gamma}{\log 2} \geqslant m - 1. \tag{8}$$

Since $\log\gamma/\log 2 = 0.40568\ldots$ we have that if $n \leqslant 500$ then $m \leqslant 204$. We ran a *Mathematica* program in the range $0 \leqslant n_1 < n \leqslant 500$, $0 \leqslant m_1 < m \leqslant 204$ and, with our conventions, we obtained all the solutions listed in Theorem 1.1.

From now on we assume $n > 500$. Thus, from (8) we have that $m > 199$ and also that $n > m$. From Binet's formula (4) we rewrite our equation as

$$|c_1\gamma^n - 2^m| < \gamma^{n_1-1} + 2\,|c_2|\,|\delta|^n + 2^{m_1} < \gamma^{n_1+3} + 2^{m_1} < \max\{\gamma^{n_1+6}, 2^{m_1+1}\}.$$

Dividing through by 2^m we get

$$\left|c_1\gamma^n 2^{-m} - 1\right| < \max\{\gamma^{n_1-n+14}, 2^{m_1-m+1}\}, \tag{9}$$

where we have used $\gamma^{n-8} < 2^m$. Let Λ be the expression inside the absolute value in the left-hand side of (9). Observe that $\Lambda \neq 0$. To see this, we consider the $\mathbb{Q}-$automorphism σ of the Galois extension $\mathbb{K} := \mathbb{Q}(\gamma, \delta)$ over $\mathbb{Q}$ defined by $\sigma(\gamma) := \delta$ and $\sigma(\delta) := \gamma$. We note $\sigma(\overline{\delta}) = \overline{\delta}$. If $\Lambda = 0$ then $\sigma(\Lambda) = 0$ and we get

$$2^m = \sigma(c_1\gamma^n) = c_2\delta^n.$$

Thus,

$$2^m = |c_2||\delta|^n < 1,$$

which is absurd since $m > 199$. Hence, $\Lambda \neq 0$. We apply Matveev's inequality to Λ by taking

$$\alpha_1 = c_1,\ \alpha_2 = \gamma,\ \alpha_3 = 2, \quad b_1 = 1,\ b_2 = n,\ b_3 = -m.$$

Thus $B = n$. Further $h(\alpha_2) = \log\gamma/3$, $h(\alpha_3) = \log 2$. For α_1 we use the properties of the height to conclude

$$h(c_1) \leqslant \log\gamma + 5\log 2.$$

So we take $A_1 = 11.3$, $A_2 = 0.3$, $A_3 = 2.1$. From Matveev's inequality we obtain

$$\log|\Lambda| > -C(1+\log n)\cdot 11.3\cdot 0.3\cdot 2.1,$$

which compared with (9) yields

$$\min\{(n-n_1)\log\gamma, (m-m_1)\log 2\} \leqslant 1.92529\times 10^{13}(1+\log n).$$

Now we study each one of these two posibilities.

Case 1. $\min\{(n-n_1)\log\gamma, (m-m_1)\log 2\} = (n-n_1)\log\gamma$.

In this case, we rewrite our equation as

$$\left|c_1(\gamma^{n-n_1}-1)\gamma^{n_1} - 2^m\right| \leqslant 2|c_2||\delta|^n + 2|c_2||\overline{\delta}|^{n_1} + 2^{m_1} < 1 + 2^{m_1} \leqslant 2^{m_1+1},$$

since $n > 500$. Thus

$$\left|c_1(\gamma^{n-n_1}-1)\gamma^{n_1}2^{-m} - 1\right| < \frac{1}{2^{m-m_1-1}}. \tag{10}$$

Let Λ_1 be the expression inside the absolute value in the left-hand side of (10). We note that $\Lambda_1 \neq 0$. For if not, we apply σ as above and we have $\sigma(\Lambda_1) = 0$. Thus

$$2^m = |\sigma(c_1)\,(\delta^n - \delta^{n_1})| \leqslant 2|c_2| < 1,$$

which is absurd since $m > 199$. We apply Matveev's inequality to Λ_1 and for this we take

$$\alpha_1 = c_1(\gamma^{n-n_1} - 1),\ \alpha_2 = \gamma,\ \alpha_3 = 2, \quad b_1 = 1,\ b_2 = n_1,\ b_3 = -m.$$

We have $B = n$. Further, the heights of α_2 and α_3 are already calculated. For α_1 we use the height properties and we get

$$\begin{aligned} h(\alpha_1) &\leqslant h(c_1) + (n-n_1)h(\gamma) + \log 2 \\ &\leqslant \frac{3\log\gamma + 18\log 2 + (n-n_1)\log\gamma}{3} \\ &\leqslant \frac{1.92530 \times 10^{13}(1+\log n)}{3}, \end{aligned}$$

where we have used (3). Thus we take $A_1 = 1.92530 \times 10^{13}(1+\log n)$, $A_2 = 0.3$, $A_3 = 2.1$. From Matveev's inequality we obtain

$$\log|\Lambda_1| > -C(1+\log n)\cdot\big(1.92530\times 10^{13}(1+\log n)\big)\cdot 0.3\cdot 2.1,$$

which compared with (10) gives us

$$(m-m_1)\log 2 < 3.28032 \times 10^{25}(1+\log n)^2.$$

Case 2. $\min\{(n-n_1)\log\gamma, (m-m_1)\log 2\} = (m-m_1)\log 2.$

In this case, we rewrite our equation as

$$\left|c_1\gamma^n - (2^{m-m_1}-1)2^{m_1}\right| \leqslant \gamma^{n_1-1} + 2|c_2||\overline{\delta}|^n \leqslant 2\gamma^{n_1} < \gamma^{n_1+3}.$$

Thus,

$$\left|1 - \left(\frac{2^{m-m_1}-1}{c_1}\right)\gamma^{-n}2^{m_1}\right| < \frac{1}{\gamma^{n-n_1-6}}, \tag{11}$$

where we have used $1 < c_1\gamma^3$. Let Λ_2 be the expression inside the absolute value in the left-hand side of (11). We note that $\Lambda_2 \neq 0$. Indeed, if it is not the case then by applying σ as above we obtain $\sigma(\Lambda_2) = 0$. Thus

$$1 \leqslant (2^{m-m_1}-1)2^{m_1} = 2^m - 2^{m_1} = |c_2||\delta|^n < |c_2| < \frac{1}{2}.$$

Now we apply Matveev's inequality to Λ_2. To do this we take

$$\alpha_1 = \frac{2^{m-m_1}-1}{c_1},\ \alpha_2 = \gamma,\ \alpha_3 = 2, \quad b_1 = 1,\ b_2 = -n,\ b_3 = m_1.$$

Thus $B = n$. The heights of α_2 and α_3 are already calculated. From the properties of the height for α_1 we obtain

$$\begin{aligned} h(\alpha_1) &\leqslant (m-m_1)h(2) + \log 2 + h(c_1) \\ &\leqslant (m-m_1)\log 2 + \log\gamma + 6\log 2 \\ &\leqslant 1.92530 \times 10^{13}(1+\log n), \end{aligned}$$

where again we use (3). Thus we take $A_1 = 5.7759 \times 10^{13}(1 + \log n)$, $A_2 = 0.3$, $A_3 = 2.1$. Thus from Matveev's inequality we obtain

$$\log |\Lambda_2| > -C(1 + \log n) \cdot \big(5.7759 \times 10^{13}(1 + \log n)\big) \cdot 0.3 \cdot 2.1,$$

which compared with (11) yields

$$(n - n_1)\log \gamma < 9.84094 \times 10^{25}(1 + \log n)^2.$$

So, from the conclusion of the two cases we have that

$$\max\{(n - n_1)\log \gamma, (m - m_1)\log 2\} < 9.84094 \times 10^{25}(1 + \log n)^2.$$

Now we get a bound on n. To do this, we rewrite our equation as

$$\left|c_1(\gamma^{n-n_1} - 1)\gamma^{n_1} - (2^{m-m_1} - 1)2^{m_1}\right| < 4|c_2||\delta|^{n_1} \leqslant 4|c_2| < 2.$$

Thus

$$\left|\frac{c_1(\gamma^{n-n_1} - 1)}{2^{m-m_1} - 1}\gamma^{n_1}2^{-m_1} - 1\right| < \frac{2}{2^m - 2^{m_1}} \leqslant \frac{4}{2^m} < \frac{4}{\gamma^{n-8}} < \frac{1}{\gamma^{n-13}}, \tag{12}$$

where we have used $\gamma^{n-8} < 2^m$ and $4 < \gamma^5$. Let Λ_3 be the expression inside the absolute value in the left-hand side of (12). If $\Lambda_3 = 0$ we apply σ as above and we obtain $\sigma(\Lambda_3) = 0$. Then

$$1 \leqslant (2^{m-m_1} - 1)2^{m_1} = 2^m - 2^{m_1} = |c_2(\delta^n - \delta^{n_1})| \leqslant 2|c_2| \leqslant \frac{2}{3},$$

which is false. Thus $\Lambda_3 \neq 0$ and we apply Matveev's inequality to it. We take

$$\alpha_1 = \frac{c_1(\gamma^{n-n_1} - 1)}{2^{m-m_1} - 1},\ \alpha_2 = \gamma,\ \alpha_3 = 2, \quad b_1 = 1,\ b_2 = n_1,\ b_3 = -m_1.$$

Thus $B = n$. The height of α_2 and α_3 have already been calculated. For α_1 we use the properties of the height and we get

$$\begin{aligned} h(\alpha_1) &\leqslant \log \gamma + (n - n_1)\frac{\log \gamma}{3} + (m - m_1)\log 2 + 7\log 2 \\ &< \frac{3.93639 \times 10^{26}(1 + \log n)^2}{3}. \end{aligned}$$

Thus we take $A_1 = 3.93639 \times 10^{26}(1 + \log n)^2$, $A_2 = 0.3$, $A_3 = 2.1$. From Matveev's inequality we get

$$\log |\Lambda_3| > -C \cdot (1 + \log n) \cdot \big(3.93639 \times 10^{26}(1 + \log n)^2\big) \cdot 0.3 \cdot 2.1,$$

which comparing with (12) we get $n < 1.90805 \times 10^{40}(\log n)^3$. Now, from Lemma 2.3 we obtain

$$n < 1.21791 \times 10^{47}.$$

Now we will reduce this upper bound on n. To do this let Γ be defined as

$$\Gamma = n\log\gamma - m\log 2 + \log c_1,$$

and we go to equation (9). We assume $\min\{n-n_1, m-m_1\} \geqslant 20$. Observe that $e^{\Gamma}-1=\Lambda\neq 0$. Therefore $\Gamma\neq 0$. If $\Gamma>0$ we have

$$0<\Gamma<e^{\Gamma}-1=|\Lambda|<\max\{\gamma^{n_1-n+14}, 2^{m_1-m+1}\}.$$

If $\Gamma<0$ we then have that $1-e^{\Gamma}=|e^{\Gamma}-1|<1/2$. Thus $e^{|\Gamma|}<2$ and we get

$$0<|\Gamma|<e^{|\Gamma|}-1=e^{|\Gamma|}|\Lambda|<2\max\{\gamma^{n_1-n+14}, 2^{m_1-m+1}\}.$$

So, in both cases we have

$$0<|\Gamma|<2\max\{\gamma^{n_1-n+14}, 2^{m_1-m+1}\}.$$

Dividing through $\log 2$ we obtain

$$0<|n\tau-m+\mu|<\max\left\{\frac{148}{\gamma^{n-n_1}}, \frac{6}{2^{m-m_1}}\right\}, \tag{13}$$

where

$$\tau:=\frac{\log\gamma}{\log 2}, \qquad \mu:=\frac{\log c_1}{\log 2}.$$

Now we apply Lemma 2.2. To do this we take $M:=1.21791\times 10^{47}$ which is the upper bound on n. Using *Mathematica* we find that the denominator of the convergent

$$\frac{p_{115}}{q_{115}}=\frac{224745259913651824657224705332045796463030735851962 6}{5539892570194379685318407717184223926861580420931369}$$

of τ is such that $q_{115}>6\,M$ and that $\varepsilon=\|q_{115}\,\mu\|-M\|q_{115}\,\tau\|=0.419327>0$. Thus with $A:=148$, $B:=\gamma$ or $A:=6$, $B:=2$ from Lemma 2.2 we obtain that, either

$$n-n_1\leqslant 444 \qquad \text{or} \qquad m-m_1\leqslant 175.$$

Now, we study each one of these two cases. First, we assume that $n-n_1\leqslant 444$ and $m-m_1\geqslant 20$. In this case we consider

$$\Gamma_1=n_1\log\gamma-m\log 2+\log\left(c_1(\gamma^{n-n_1}-1)\right),$$

and we go to (10). We see that $e^{\Gamma_1}-1=\Lambda_1\neq 0$. Thus $\Gamma_1\neq 0$ and, with a similar argument as the previous one we obtain

$$0<|\Gamma_1|<\frac{4}{2^{m-m_1}}.$$

Divinding through $\log 2$ we get

$$0 < |n_1\tau - m + \mu| < \frac{6}{2^{m-m_1}}, \tag{14}$$

where τ is the same one as above and

$$\mu := \frac{\log\left(c_1(\gamma^{n-n_1} - 1)\right)}{\log 2}.$$

We note that $n_1 > 0$, since otherwise we would have $n \leqslant 444$ which contradicts $n > 500$. Thus we can apply Lemma 2.2. Consider

$$\mu_k := \frac{\log\big(c_1(\gamma^k - 1)\big)}{\log 2}, \qquad k = 1, 2, \ldots, 444.$$

With the help of *Mathematica* we found that the same 115-th convergent of τ well works, that is, it is such that its denominator satisfies $q_{115} > 6\,M$ and $\varepsilon_k \geqslant 0.000889789 > 0$ for all $k = 1, 2, \ldots, 444$. Thus, with $A := 6$, $B := 2$ we calculated $\log(q_{115} \cdot 6/\varepsilon_k)/\log 2$ for all $k = 1, 2, \ldots, 444$ and we found that the maximum value of them is at most 184. Therefore, $m - m_1 \leqslant 184$.

We now study the other case. Assume $m - m_1 \leqslant 175$ and $n - n_1 \geqslant 20$. Consider

$$\Gamma_2 = n \log\gamma - m_1 \log 2 + \log\left(\frac{c_1}{2^{m-m_1} - 1}\right),$$

and we go to (11). We see that $1 - e^{-\Gamma_2} = \Lambda_2 \neq 0$. Thus $\Gamma_2 \neq 0$. As before, we can deduce that

$$0 < |\Gamma_2| < \frac{2\gamma^6}{\gamma^{n-n_1}}.$$

Dividing through $\log 2$ we get

$$0 < |n\tau - m_1 + \mu| < \frac{16}{\gamma^{n-n_1}}, \tag{15}$$

where τ is as above and

$$\mu := \frac{\log\left(c_1/(2^{m-m_1} - 1)\right)}{\log 2}.$$

Note that $m_1 > 0$. For if not, then $m \leqslant 175$ which contradicts $m > 199$. Thus we can apply Lemma 2.2 again. Consider

$$\mu_\ell := \frac{\log\big(c_1/(2^\ell - 1)\big)}{\log 2} \qquad \ell = 1, \ldots, 175.$$

With *Mathematica*, we find that again the same 115-th convergent of τ above well works, that is, $q_{115} > 6\,M$ and $\varepsilon_\ell \geqslant 0.0016923 > 0$ for all $\ell = 1, 2, \ldots, 175$.

Thus, with $A := 16$, $B := \gamma$ we calculated $\log(q_{115} \cdot 16/\varepsilon_\ell)/\log\gamma$ for all $\ell = 1, 2, \ldots, 175$ and found that the maximum of its values is $\leqslant 456$. Thus $n - n_1 \leqslant 456$.

Let us summarize what we have done. We got that either $n - n_1 \leqslant 444$ or $m - m_1 \leqslant 175$. Assuming the first one, we got that $m - m_1 \leqslant 184$ and, assuming the second one we got $n - n_1 \leqslant 456$. Altogether, we have that $n - n_1 \leqslant 456$ and $m - m_1 \leqslant 184$. So, it remains to study this case.

To do this, we consider

$$\Gamma_3 = n_1 \log\gamma - m_1 \log 2 + \log\left(\frac{c_1(\gamma^{n-n_1} - 1)}{2^{m-m_1} - 1}\right),$$

and we go to (12). Note that $e^{\Gamma_3} - 1 = \Lambda_3 \neq 0$. Thus $\Gamma_3 \neq 0$. As above we can deduce

$$0 < |\Gamma_3| < \frac{2 \cdot \gamma^{13}}{\gamma^n},$$

since $n > 500$. Dividing through by $\log 2$ we get

$$0 < |n_1\tau - m_1 + \mu| < \frac{112}{\gamma^n}, \tag{16}$$

where τ is as above and

$$\mu := \frac{\log\left(c_1(\gamma^{n-n_1} - 1)/(2^{m-m_1} - 1)\right)}{\log 2}.$$

As before, we note that $n_1 > 0$ and $m_1 > 0$. Thus we can apply Lemma 2.2 again. Consider

$$\mu_{k,\ell} := \frac{\log\big(c_1(\gamma^k - 1)/(2^\ell - 1)\big)}{\log 2}, \quad k = 1, \ldots, 456, \quad \ell = 1, \ldots, 184.$$

With the help of *Mathematica* we find again that the same 115-th convergent above of τ works also well in this case, that is, $q_{115} > 6\,M$ and $\varepsilon_{k,\ell} \geqslant 5.27725 \times 10^{-6} > 0$ for all $k = 1, \ldots, 456$, $\ell = 1, \ldots, 184$. Thus, with $A := 112$, $B := \gamma$ we calculated $\log(q_{115} \cdot 112/\varepsilon_{k,\ell})/\log\gamma$, for all $k = 1, \ldots, 456$, $\ell = 1, \ldots, 184$ and found that the maximum value of them is $\leqslant 483$. Thus $n \leqslant 483$, which contradicts our assumption on n. This finishes the proof of Theorem 1.1.

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