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Sandwich theorem for reciprocally strongly convex functions

Teorema del Sandwich para funciones fuerte-recíprocamente convexas

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Abstract. We introduce the notion of reciprocally strongly convex functions and we present some examples and properties of them. We also prove that two real functions f and g, defined on a real interval $[a,b]$, satisfy

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tg\left(y\right) + (1-t)g\left(x\right) - ct(1-t)\left(\frac{1}{x}-\frac{1}{y}\right)^2,$$

for all $x, y \in [a,b]$ and $t \in [0,1]$ iff there exists a reciprocally strongly convex function $h : [a,b] \to \mathbb{R}$ such that $f\left(x\right) \leq h\left(x\right) \leq g\left(x\right)$ for all $x \in [a,b]$.

Finally, we obtain an approximate convexity result for reciprocally strongly convex functions; namely we prove a stability result of Hyers-Ulam type for this class of functions.

Key words and phrases. Convex functions,Sandwich theorem, Hyers-Ulam.

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Resumen. En este artículo introducimos la noción de funciones recíproca-fuertemente convexas y presentamos algunos ejemplos y propiedades. Además se demuestran que dos funciones f y g, definidas en el intervalo real $[a,b]$ satisfacen la desigualdad

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tg\left(y\right) + (1-t)g\left(x\right) - ct(1-t)\left(\frac{1}{x}-\frac{1}{y}\right)^2,$$

para todo $x, y \in [a, b]$ y $t \in [0, 1]$ si, y sólo si, existe una función recíproca-fuertemente convexa $h : [a, b] \to \mathbb{R}$ tal que $f(x) \leq h(x) \leq g(x)$ para todo $x \in [a, b]$.

Finalmente, se obtiene un resultado de aproximación convexa para esta clase de funciones.

Palabras y frases clave. Funciones convexas, Teorema del Sandwich, Hyers-Ulam.

1. Introduction

Due to its important role in mathematical economics, engineering, management science and optimization theory, convexity of functions and sets has been studied intensively; see [3, 7, 9, 11, 12, 13, 19, 22, 23] and the references therein. Consequently, the classical concepts of convex functions has been extended and generalized in different directions.

The most important generalizations can be found in works in which, by changing the way of defining the functions, one obtains generalizations such as quasi-convex (see [10]), pseudo-convex (see [1]), strongly convex [29], approximately convex [6]. This midconvex (see [30]), h-convex functions [33], etc.

In this article, we deal with a recent notion of generalized convexity. This notion was introduced by I. Iscan in [19] Iscan gave the following definition of harmonically convex functions:

Definition 1.1 ([19])**.** Let I be an interval in $\mathbb{R} \setminus \{0\}$. A function $f : I \to \mathbb{R}$ is said to be *harmonically convex* on I if the inequality

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf(y) + (1-t)f(x), \tag{1}$$

holds, for all $x, y \in I$ and $t \in [0, 1]$.

For some recent results and extensions of harmonically convex functions, the interested readers are referred to [8, 17, 18, 19, 27, 28, 34].

In [8] we can find the following simple but important fact:

Theorem 1.2 ([8])**.** *If $[a, b] \subset I \subseteq (0, +\infty)$ and if we consider the function $g : \left[\frac{1}{b}, \frac{1}{a}\right] \to \mathbb{R}$, defined by $g(t) = f\left(\frac{1}{t}\right)$, then f is harmonically convex on $[a, b]$ if and only if g is convex in the usual sense on $\left[\frac{1}{b}, \frac{1}{a}\right]$.*

It is easy to verify that this result is satisfied if we use the interval $(0, +\infty)$ rather than the interval $[a, b]$.

The following theorem on separation of functions (a *sandwich theorem*) can be found in the seminal papers of Baron et.al. [2].

Theorem 1.3 ([2])**.** *Two real-valued functions f and g, defined on a real interval I satisfy*
$$f(tx+(1-t)y) \leq tg(x)+(1-t)g(y) \tag{2}$$
for all $x, y \in I$ and $t \in [0,1]$, if and only if there exists a convex function $h : I \to \mathbb{R}$ such that $f \leq h \leq g$.

2. Reciprocally strongly convex functions

In 1966 Polyak [29] introduced the notions of strongly convex and strongly quasi-convex functions. In 1976 Rockafellar [31] studied the strongly convex functions in connection with *the proximal point algorithm.* They play an important role in optimization theory and mathematical economics. Also, Nikodem et al. have obtained some interesting properties of strongly convex functions (see [9, 14, 20]).

Definition 2.1 (See [14, 23, 30])**.** Let D be a convex subset of $\mathbb{R}$ and let $c > 0$. A function $f : D \to \mathbb{R}$ is called strongly convex with modulus c if
$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)-ct(1-t)(x-y)^2 \tag{3}$$
for all $x, y \in D$ and $t \in [0,1]$.

The usual notion of convex function correspond to the case $c = 0$. For instance, if f is strongly convex, then it is bounded from below, its level sets $\{x \in I : f(x) \leq \lambda\}$ are bounded for each λ and f has a unique minimum on every closed subinterval of I ([25], p. 268). Any strongly convex function defined on a real interval admits a quadratic support at every interior point of its domain.

The proofs of the next two theorems can be found in [30].

Theorem 2.2. *Let D be a convex subset of $\mathbb{R}$ and let c be a positive constant. A function $f : D \to \mathbb{R}$ is strongly convex with modulus c if and only if the function $g(x) = f(x) - cx^2$ is convex.*

Theorem 2.3. *The following are equivalent:*

(1) $f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)-t(1-t)c(x-y)^2$, *for all $x, y \in (a,b)$ and $t \in [0,1]$.*

(2) For each $x_0 \in (a,b)$, there is a linear function T such that $f(x) \geq f(x_0)+T(x-x_0)+c(x-x_0)^2$ *for all $x, y \in (a,b)$.*

(3) For differentiable f, for each $x_0 \in (a,b)$: $f(x) \geq f(x_0)+f'(x_0)(x-x_0)+c(x-x_0)^2$, *for all $x \in (a,b)$.*

(4) For twice differentiable f, $f''(x) \geq 2c$, for all $x \in (a,b)$.

In [5] we proved the following sandwich theorem for harmonically convex functions:

Theorem 2.4. *Let f, g be real functions defined on the interval $(0, +\infty)$. The following conditions are equivalent:*

(i) there exists a harmonically convex function $h : (0, +\infty) \to \mathbb{R}$ such that $f(x) \leq h(x) \leq g(x)$, for all $x \in (0, +\infty)$;

(ii) the following inequality holds

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq tg(y) + (1-t)g(x), \tag{4}$$

for all $x, y \in (0, +\infty)$ and $t \in [0, 1]$.

On the other hand, in [4], we introduced the notion of harmonically strongly convex function, as follows:

Definition 2.5. Let I be an interval in $\mathbb{R} \setminus \{0\}$ and let $c \in \mathbb{R}_+$. A function $f : I \to \mathbb{R}$ is said to be *harmonically strongly convex with modulus c* on I, if the inequality

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf(y) + (1-t)f(x) - ct(1-t)(x-y)^2, \tag{5}$$

holds, for all $x, y \in I$ and $t \in [0, 1]$.

The symbol $\mathrm{SHC}_{(I,c)}$ will denote the class of functions that satisfy the inequality (5). We also establish some Hermite-Hadamard and Fejér type inequalities for the class of harmonically strongly convex functions.

Next we will explore a generalization of the concept of harmonically convex functions which we will call reciprocally strongly convex functions, it is a concept parallel to the definition presented in the definition 2.5.

Definition 2.6. Let I be an interval in $\mathbb{R} \setminus \{0\}$ and let $c \in (0, \infty)$. A function $f : I \to \mathbb{R}$ is said to be *reciprocally strongly convex* with modulus c on I, if the inequality

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf(y) + (1-t)f(x) - ct(1-t)\left(\frac{1}{x} - \frac{1}{y}\right)^2, \tag{6}$$

holds, for all $x, y \in I$ and $t \in [0, 1]$.

The symbol $\mathrm{SRC}_{(I,c)}$ will denote the class of functions that satisfy the inequality (6).

Theorem 2.7. *Let $I \subset \mathbb{R}\backslash\{0\}$ be a real interval and $c \in (0,\infty)$. If $f \in SRC_{(I,c)}$, then f is harmonically convex.*

Proof. Since $ct(1-t)\left(\frac{1}{x}-\frac{1}{y}\right)^2 \geq 0$, it is an immediate consequence of the definition. ☑

For the rest of this paper, we will use $I \subset \mathbb{R} \setminus \{0\}$ to denote a real interval and $c \in (0,\infty)$.

Theorem 2.8. *Let $f : I \to \mathbb{R}$ be a function. $f \in SRC_{(I,c)}$ if and only if the function $g : I \to \mathbb{R}$, defined by $g(x) := f(x) - \frac{c}{x^2}$ is harmonically convex.*

Proof. Assume that $f \in \mathrm{SRC}_{(I,c)}$, then

$$
\begin{aligned}
&g\left(\frac{xy}{tx+(1-t)y}\right)\\
&= f\left(\frac{xy}{tx+(1-t)y}\right) - c\left(\frac{tx+(1-t)y}{xy}\right)^2\\
&\leq tf(y)+(1-t)f(x) - ct(1-t)\left(\frac{1}{y}-\frac{1}{x}\right)^2 - c\left(t\frac{1}{y}+(1-t)\frac{1}{x}\right)^2\\
&= tf(y)+(1-t)f(x) - ct(1-t)\left(\frac{1}{y^2}-\frac{2}{xy}+\frac{1}{x^2}\right)\\
&-c\left(\frac{t^2}{y^2}+\frac{2t(1-t)}{xy}+\frac{(1-t)^2}{x^2}\right)\\
&= tf(y)+(1-t)f(x) - c\left(\frac{t}{y^2}-\frac{2t}{xy}+\frac{t}{x^2}-\frac{t^2}{y^2}+\frac{2t^2}{xy}-\frac{t^2}{x^2}\right.\\
&\left.+\frac{t^2}{y^2}+\frac{2t}{xy}-\frac{2t^2}{xy}+\frac{1}{x^2}-\frac{2t}{x^2}+\frac{t^2}{x^2}\right)\\
&= tf(y)+(1-t)f(x) - c\left(\frac{t}{y^2}+\frac{1}{x^2}-\frac{t}{x^2}\right)\\
&= tf(y)+(1-t)f(x) - c\left(\frac{t}{y^2}+(1-t)\frac{1}{x^2}\right)\\
&= t\left(f(y)-\frac{c}{y^2}\right)+(1-t)\left(f(x)-\frac{c}{x^2}\right)\\
&= tg(y)+(1-t)g(x),
\end{aligned}
$$

for all $x, y \in I$ and $t \in [0,1]$, which proves that g is harmonically convex.

Conversely, if g is harmonically convex, then

$$
\begin{aligned}
& f\left(\frac{xy}{tx+(1-t)y}\right) \\
& = g\left(\frac{xy}{tx+(1-t)y}\right) + c\left(\frac{tx+(1-t)y}{xy}\right)^2 \\
& \leq tg(y)+(1-t)g(x)+c\left(t\frac{1}{y}+(1-t)\frac{1}{x}\right)^2 \\
= {} & tg(y)+(1-t)g(x)+c\left(\frac{t^2}{y^2}+\frac{2t(1-t)}{xy}+\frac{(1-t)^2}{x^2}\right) \\
= {} & tg(y)+(1-t)g(x)+c\left(\frac{t(1-1+t)}{y^2}+\frac{2t(1-t)}{xy}+\frac{(1-t)(1-t)}{x^2}\right) \\
= {} & tg(y)+(1-t)g(x)+c\left(\frac{t(1-1+t)}{y^2}+\frac{2t(1-t)}{xy}+\frac{(1-t)(1-t)}{x^2}\right) \\
= {} & tg(y)+(1-t)g(x)+c\left(\frac{t}{y^2}-\frac{t(1-t)}{y^2}+\frac{2t(1-t)}{xy}+\frac{1-t}{x^2}-\frac{t(1-t)}{x^2}\right) \\
= {} & t\left(g(y)+c\frac{1}{y^2}\right)+(1-t)\left(g(x)+c\frac{1}{x^2}\right)-ct(1-t)\left(\frac{1}{y^2}-\frac{2}{xy}+\frac{1}{x^2}\right) \\
= {} & tf(y)+(1-t)f(x)-ct(1-t)\left(\frac{1}{y}-\frac{1}{x}\right)^2,
\end{aligned}
$$

for all $x, y \in I$ and $t \in [0,1]$, showing that $f \in \mathrm{SRC}_{(I,c)}$. ☑

Example 2.9. (1) The constant function is harmonically convex but not reciprocally strongly convex.

(2) The function $f : (0,+\infty) \to \mathbb{R}$ defined by $f(x) = -x^2$, is not a harmonically convex function, since f is a not convex and nonincreasing function. Based on Theorem 2.7, we obtain $f \notin \mathrm{SRC}_{(I,c)}$.

(3) Since $g(x) = \log(x)$ is a harmonically convex function, the function $f(x) := \log(x) + \dfrac{c}{x^2}$ is a reciprocally strongly convex function.

Lemma 2.10. *If f is a reciprocally strongly convex function, then the function $\varphi = f + \epsilon$ is also a reciprocally strongly convex function, for any constant ϵ. In fact,*

$$
\begin{aligned}
\varphi\left(\frac{xy}{tx+(1-t)y}\right) &= f\left(\frac{xy}{tx+(1-t)y}\right)+\epsilon \\
&\le tf(y)+(1-t)f(x)+ct(1-t)\left(\frac{1}{x}-\frac{1}{y}\right)^2+\epsilon \\
&= tf(y)+t\epsilon+(1-t)f(x)+(1-t)\epsilon+ct(1-t)\left(\frac{1}{x}-\frac{1}{y}\right)^2 \\
&= t(f(y)+\epsilon)+(1-t)(f(x)+\epsilon)+ct(1-t)\left(\frac{1}{x}-\frac{1}{y}\right)^2 \\
&= t\varphi(y)+(1-t)\varphi(x)+ct(1-t)\left(\frac{1}{x}-\frac{1}{y}\right)^2.
\end{aligned}
$$

Theorem 2.11. *If $f:[a,b]\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ and if we consider the function $g:\left[\frac{1}{b},\frac{1}{a}\right]\to\mathbb{R}$, defined by $g(t)=f\left(\frac{1}{t}\right)$, then $f\in SRC_{([a,b],c)}$ if and only if g is strongly convex in $\left[\frac{1}{b},\frac{1}{a}\right]$.*

Proof. If for all $x,y\in[a,b]$ and $t\in[0,1]$, we have

$$
f\left(\frac{1}{t\frac{1}{y}+(1-t)\frac{1}{x}}\right)\le tf(y)+(1-t)f(x)-ct(1-t)\left(\frac{1}{x}-\frac{1}{y}\right)^2;
$$

this last inequality may be changed by another equivalent one:

$$
g\left(tw+(1-t)u\right)\le tg\left(w\right)+(1-t)g\left(u\right)-ct(1-t)\left(u-w\right)^2,
$$

where $u,w\left[\frac{1}{b},\frac{1}{a}\right]$ and $t\in[0,1]$. This completes the proof. ☑

It is easy to see that the result is also valid for intervals $(a,b)\subset\mathbb{R}\setminus\{0\}$.

Theorem 2.12. *The following are equivalent:*

(i) $f\in SRC_{((a,b),c)}$.

(ii) For each $x_0\in(a,b)$, there is a linear function T such that

$$
f\left(\frac{1}{x}\right)\ge c(x-x_0)^2+T(x-x_0)+f\left(\frac{1}{x_0}\right), \tag{7}
$$

for all $x\in\left(\frac{1}{b},\frac{1}{a}\right)$.

(iii) For differentiable f and $x_0 \in \left(\frac{1}{b}, \frac{1}{a}\right)$,

$$f\left(\frac{1}{x}\right) \geq f\left(\frac{1}{x_0}\right) - f'\left(\frac{1}{x_0}\right)\frac{x-x_0}{x^2} + c(x-x_0)^2, \tag{8}$$

for all $x \in \left(\frac{1}{b}, \frac{1}{a}\right)$.

(iv) For twice differentiable f,

$$\frac{1}{x^4}\left[f''\left(\frac{1}{x}\right) + 2xf'\left(\frac{1}{x}\right)\right] \geq 2c, \quad \textit{for all} \quad x \in \left(\frac{1}{b}, \frac{1}{a}\right).$$

Proof. [(i)⇒ (ii)] Assume that $f \in \mathrm{SRC}_{((a,b),c)}$. Since all the assumptions of Theorem 2.11 are satisfied, then the function $g(x) := f\left(\frac{1}{x}\right)$ is strongly convex in $\left(\frac{1}{b}, \frac{1}{a}\right)$. Then by Theorem 2.3, for each $x_0 \in \left(\frac{1}{b}, \frac{1}{a}\right)$, there is a linear function T such that $g(x) \geq g(x_0) + T(x-x_0) + c(x-x_0)^2$, for all $x, y \in \left(\frac{1}{b}, \frac{1}{a}\right)$. This is equivalent to the inequality (7).

[(i)⇒ (iii)] Assume that $f \in \mathrm{SRC}_{((a,b),c)}$. By Theorem 2.11, the function $g(x) := f\left(\frac{1}{x}\right)$ is strongly convex in $\left(\frac{1}{b}, \frac{1}{a}\right)$, then by Theorem 2.3, for each $x_0 \in \left(\frac{1}{b}, \frac{1}{a}\right)$ $g(x) \geq g(x_0) + g'(x_0)(x-x_0) + c(x-x_0)^2$, for all $x, y \in (a,b)$. This is equivalent to the inequality (8).

(ii)⇒ (i) and (iii)⇒ (i) are shown using the reciprocals of the theorem and lemma that we have used in the above part.

[(i)⇔ (iv)] Suppose f is twice differentiable over (a,b). $f \in \mathrm{SRC}_{((a,b),c)}$ if only if the function $g(x) := f\left(\frac{1}{x}\right)$ is strongly convex in $\left(\frac{1}{b}, \frac{1}{a}\right)$ (by the theorem 2.11). It follows from Theorem 2.3 that g is a strongly convex function with modulus c if only if $g''(x) \geq 2c$. Hence it is equivalent to

$$\frac{1}{x^4}\left[f''\left(\frac{1}{x}\right) + 2xf'\left(\frac{1}{x}\right)\right] \geq 2c, \quad \text{for all} \quad x \in \left(\frac{1}{b}, \frac{1}{a}\right).$$

☑

3. Main results

In this section, we derive our main results.

Theorem 3.1 (Sandwich theorem)**.** *Let $I \subset (0, +\infty)$ be an interval and let $f, g : I \to \mathbb{R}$. The following conditions are equivalent:*

(i) there exists a reciprocally strongly convex function with modulus c, $h : I \to \mathbb{R}$ such that $f(x) \le h(x) \le g(x)$, for all $x \in I$;

(ii) the following inequalities hold

$$f\left(\frac{xy}{tx+(1-t)y}\right) \le tg(y) + (1-t)g(x) - ct(1-t)\left(\frac{1}{x} - \frac{1}{y}\right)^2, \quad (9)$$

for all $x, y \in I,\ t \in [0,1]$.

Proof. [(i)⇒(ii)] Assume that there exists a reciprocally strongly convex function with modulus c, $h : I \to \mathbb{R}$ such that $f(x) \le h(x) \le g(x)$, for all $x \in I$.

Then,

$$\begin{aligned} f\left(\frac{xy}{tx+(1-t)y}\right) &\le h\left(\frac{xy}{tx+(1-t)y}\right) \\ &\le th(y) + (1-t)h(x) - ct(1-t)\left(\frac{1}{x} - \frac{1}{y}\right)^2 \\ &\le tg(y) + (1-t)g(x) - ct(1-t)\left(\frac{1}{x} - \frac{1}{y}\right)^2, \end{aligned}$$

for all $x, y \in I$ and $t \in [0,1]$, as desired.

[(ii)⇒(i)] Conversely, if (9) holds, we define the functions $f_1, g_1 : I \to \mathbb{R}$

$$f_1(x) := f(x) - \frac{c}{x^2} \ \text{ and } \ g_1(x) := g(x) - \frac{c}{x^2}.$$

Consequently,

$$\begin{aligned} f_1\left(\frac{xy}{tx+(1-t)y}\right) &= f\left(\frac{xy}{tx+(1-t)y}\right) - c\left(\frac{tx+(1-t)y}{xy}\right)^2 \\ &\le tg(y)+(1-t)g(x)-ct(1-t)\left(\frac{1}{x}-\frac{1}{y}\right)^2 - c\left(\frac{t}{y}+\frac{1-t}{x}\right)^2 \\ &= tg(y)+(1-t)g(x)-t\frac{c}{y^2}-(1-t)\frac{c}{x^2} = tg_1(y)+(1-t)g_1(x), \end{aligned}$$

for all $x, y \in I$ and $t \in [0,1]$. By Theorem 2.4 there exists a harmonically convex function $h_1 : I \to \mathbb{R}$ such that $f_1 \le h_1 \le g_1$ on I. By Theorem 2.8, the function $h(x) := h_1(x) + \frac{c}{x^2}$ is reciprocally strongly convex and satisfies

$$f(x) \le h(x) \le g(x),$$

for all $x \in I$. ☑

Corollary 3.2. *If f, g_1, g_2 are real functions defined on the interval $(0, +\infty)$ satisfying the inequality*

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq tg_1(y) + (1-t)g_2(x),$$

for all $x, y \in (0, +\infty)$ and $t \in [0,1]$ then there exists a reciprocally strongly convex function $h : (0, +\infty) \to \mathbb{R}$ such that $f(x) \leq h(x) \leq \max\{g_1, g_2\}(x)$, for all $x \in (0, +\infty)$.

The Hyers-Ulam stability problem of functional equations was originated by Ulam in 1940 when he proposed the following question [32]: Let f be a mapping from a group G_1 to a metric group G_2 with metric $d(\cdot, \cdot)$ such that

$$d(f(xy), f(x)f(y)) \leq \epsilon, x, y \in G_1.$$

Does there exist a group homomorphism h and $\delta_\epsilon > 0$ such that $d(f(x), h(x)) \leq \delta_\epsilon$, $x \in G_1$?

One of the first assertions to be obtained is the following result, due to Hyers [15], that gives an answer to the question of Ulam.

Theorem 3.3. *Suppose that S is an additive semigroup, Y is a Banach space, $\epsilon \geq 0$, and $f : S \to Y$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon, \textit{for all } x, y \in S. \tag{10}$$

Then there exists a unique function $A : S \to Y$ satisfying $A(x+y) = A(x) + A(y)$ for which $\|f(x) - A(x)\| \leq \epsilon$ for all $x \in S$.

Since then, stability problems have been investigated in various directions for many other functional equations [21].

The investigation of approximate convexity started with the paper by Hyers and Ulam [16] who in the year 1952 introduced and investigated ϵ-convex functions: If D is a convex subset of a real linear space X and ϵ is a nonnegative number, then a function $f : D \to \mathbb{R}$ is called ϵ-convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \epsilon, x, y \in D, t \in [0,1].$$

Hyers and Ulam [16] proved that any $\epsilon-$convex function on a finite dimensional convex set can be approximated by a convex function.

As an immediate consequence of Theorem 3.1 we obtain the following stability result of Hyers-Ulam type for reciprocally strongly convex functions (see [24, 26]).

Theorem 3.4. *Let $[a, b] \subseteq (0, +\infty)$ be an interval and $\epsilon > 0$. A function $f : [a, b] \to \mathbb{R}$ satisfies the inequality*

$$\left|f\left(\frac{xy}{tx + (1-t)y}\right) - tf(y) - (1-t)f(x)\right| \leq \epsilon, \tag{11}$$

for all $x, y \in [a,b]$ *and* $t \in [0,1]$ *then there exists an reciprocally strongly convex function* $\varphi : [a,b] \to \mathbb{R}$ *such that*

$$|f(x) - \varphi(x)| \leq \frac{\epsilon}{2}, \qquad x \in [a,b].$$

Proof. Note that since theorem (3.1) holds with $g = f + \epsilon$, it follows that there exists a reciprocally strongly convex function $h : [a,b] \to \mathbb{R}$ such that

$$f(x) \leq h(x) \leq f(x) + \epsilon,$$

for $x \in [a,b]$.

Defining $\varphi : [a,b] \to \mathbb{R}$ by $\varphi(x) := h(x) - \frac{\epsilon}{2}$, we obtain a reciprocally strongly convex function such that

$$f(x) - \frac{\epsilon}{2} \leq h(x) - \frac{\epsilon}{2} \leq f(x) + \frac{\epsilon}{2},$$

for all $x \in [a,b]$; that is

$$-\frac{\epsilon}{2} \leq \varphi(x) - f(x) \leq \frac{\epsilon}{2}$$

or

$$|\varphi(x) - f(x)| \leq \frac{\epsilon}{2}$$

for all $x \in [a,b]$. ☑

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References

[1] M. Avriel, W.T. Diewert, S. Schaible, and I. Zang, *Generalized concavity*, 1998.

[2] K. Baron, J. Matkowski, and K. Nikodem, *A sandwich with convexity*, Math. Pannica 5/1 (1994), 139–144.

[3] M. Bessenyei and Zs. Páles, *Characterization of convexity via Hadamard's inequality*, Math. Inequal. Appl. 9 (2006), 53–62.

[4] M. Bracamonte, J. Giménez, and J. Medina, *Hermite-Hadamard and Fejér type inequalities for strongly harmonically convex functions*, Submitted for publication (2016).

[5] M. Bracamonte, J. Giménez, J. Medina, and M. Vivas, *A sandwich theorem and stability result of Hyers-Ulam type for harmonically convex functions*, Submitted for publication (2016).

[6] A. Daniilidis and P. Georgiev, *Approximately convex functions and approximately monotonic operators*, Nonlin. Anal., 66 (2007), 547–567.

[7] S. Dragomir, *Inequalities of Hermite-Hadamard type for h-convex functions on linear spaces*, Mathematica Moravica (2015), 107–121.

[8] ______, *Inequalities of Jensen type for HA-convex functions*, RGMIA Monographs, Victoria University (2015).

[9] S. Dragomir and C. Pearce, *Selected topics on Hermite-Hadamard inequalities and applications*, 2000.

[10] A. Eberhard and Pearce C. E. M., *Class inclusion properties for convex functions, in progress in optimization*, Appl. Optim, Kluwer Acad. Publ., Dordrecht, 39 (1998), 129–133.

[11] L. Fejér, *Über die fourierreihen*, II, Math. Naturwiss. Anz Ungar. Akad.Wiss. (1906), 369–390.

[12] A. Ghazanfari and S. Dragomir, *Hermite-Hadamard inequality for functions whose derivatives absolute values are preinvex*, 2012.

[13] G.H. Hardy, J.E. Littlewood, and G. Polya, *Inequalities*, Cambridge Univ. Press., 1934.

[14] J. Hiriart-Urruty and C. Lemaréchal, *Fundamentals of convex analysis*, Springer-Verlag, Berlin-Heidelberg, 2001.

[15] D. H. Hyers, *On the stability of the linear functional equations*, Proc. Nat. Acad. Sci. USA 27 (1941), 222–224.

[16] D. H. Hyers and S. M. Ulam, *Approximately convex functions*, Proc. Amer. Math. Soc.3 (1952), 821–828.

[17] İ. Işcan, *Hermite-Hadamard type inequalities for harmonically* (α, m)-*convex functions*, Contemp. Anal. Appl. Math., 1 (2) (2013), 253–264.

[18] I. Işcan, *New estimates on generalization of some integral inequalities for s-convex functions and their applications*, Int. J. Pure Appl. Math., 86 (4) (2013), 727–726.

[19] İ. Işcan, *Hermite-Hadamard type inequalities for harmonically convex functions*, Hacettepe Journal of Mathematics and Statistics Volume 43 (6) (2014), 935 – 942.

[20] M.V. Jovanovič, *A note on strongly convex and strongly quasiconvex functions*, Notes 60 (1996), 778–779.

[21] S.M. Jung, *Hyers-ulam-rassias stability of functional equations in mathematical analysis*, Hadronic Press, Inc., Palm Harbor, 2001.

[22] M. Kuczma, *An introduction to the theory of functional equations and inequalities, Cauchy's equation and Jensen's inequality*, Second Edition, Birkhäuser, Basel Boston Berlin, 2009.

[23] N. Merentes and K. Nikodem, *Remarks on strongly convex functions*, Aequationes mathematicae,Volume 80, Issue 1 (2010), 193–199.

[24] F. C. Mitroi-Symeonidis, *Convexity and sandwich theorems*, European Journal of Research in Applied Sciences, Vol. 1, No. 1 (2015), 9–11.

[25] C. Niculescu and L. Persson, *Convex functions and their applications*, A Contemporary Approach, CMS Books in Mathematics, vol. 23, Springer, New York, 2006.

[26] K. Nikodem and S. Wąsowicz, *A sandwich theorem and Hyers-Ulam stability of affine functions*, Aequationes Math. 49 (1995), 160–164.

[27] M. A. Noor, K. I. Noor, and M. U. Awan, *Some characterizations of harmonically log-convex functions*, Proc. Jangjeon Math. Soc., 17(1) (2014), 51–61.

[28] ______, *Some integral inequalities for harmonically logarithmic h-convex functions*, preprint (2014).

[29] B. Polyak, *Existence theorems and convergence of minimizing sequences in extremum problems with restrictions*, Dokl. Akad. Nauk. SSSR 166 (1966), 287–290.

[30] A. Roberts and D. Varberg, *Convex functions*, Academic Press, New York-London, 1973.

[31] R.T. Rockafellar, *Monotone operator and the proximal point algorithm*, SIAM J. Control Optim 14 (1976), 888–898.

[32] S.M. Ulam, *A collection of mathematical problems*, Interscience Publ., New York, 1960.

[33] S. Varošanec, *On h-convexity*, J. Math. Anal. Appl. 326 (2007), 303 – 311.

[34] T.-Y Zhang, A.-P. Ji, and F. Qi, *Integral inequalities of Hermite-Hadamard type for harmonically quasiconvex functions*, Proc. Jangjeon Math. Soc., 16(3) (2013), 399–407.

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