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A Characterization of Strongly Dependent Ordered Abelian Groups

Una caracterización de los grupos abelianos fuertemente dependientes

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Abstract. We characterize all ordered Abelian groups whose first-order theory in the language $\{+, <, 0\}$ is strongly dependent. The main result of this note was obtained independently by Halevi and Hasson [7] and Farré [5].

Key words and phrases. Strongly dependent theories, NIP, ordered Abelian groups.

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Resumen. Damos una caracterización completa de los grupos abelianos ordenados cuyas teorías completas en el lenguaje $\{+, <, 0\}$ son fuertamente dependientes. El resultado principal de este artículo fue obtenido de manera independiente por Halevi y Hasson [7] y Farré [5].

Palabras y frases clave. Teorías dependientes, grupos abelianos ordenados.

1. Introduction

Given a general model-theoretic notion which may be interpreted as indicating that a structure or theory is "tame," such as stability or the absence of the independence property (see [9]), it is natural to attempt to characterize within a general class of algebraic objects which of the structures satisfy the tameness condition. In this note we carry out this program for the tameness conditions given by strong dependence and finite dp-rank within the algebraic setting of ordered Abelian groups.

One of the most important classes of tame theories is the class of NIP theories (also known as "dependent theories"). It is already known that *any* complete theory of an ordered Abelian group in $\mathcal{L}_{oag} = \{+, <, 0\}$ is NIP (see [6]), and on the other hand any ordered structure must be unstable.

The most prominent dividing line within unstable NIP theories is given by the class of *strongly NIP* (or "strongly dependent") theories, which we will now define.

Definition 1.1. A theory T in a language $\mathcal{L}$ *admits an inp-pattern of depth* κ (for κ a cardinal) if we may find a sequence $\{\varphi_i(x, \overline{y}_i) : i \in \kappa\}$ of $\mathcal{L}$-formulas, a model $\mathfrak{M}$ of T, and parameters $\overline{a}_{ij}$ for $(i, j) \in \kappa \times \omega$ in M (where $|\overline{a}_{ij}| = |\overline{y}_i|$ for all j) so that:

(1) For every $i < \kappa$, there is a $k_i \in \omega$ such that

$$\{\varphi_i(x; \overline{a}_{ij}) : j \in \omega\}$$

is k_i-inconsistent (that is, the conjunction of any k_i formulas in the set is inconsistent); and

(2) for every $\eta : \kappa \to \omega$, the set

$$\{\varphi_i(x; \overline{a}_{i,\eta(i)}) : 1 \leq i \leq n\}$$

is consistent.

If T is NIP, then T is *strongly dependent* if it does not admit an inp-pattern of depth $\aleph_0$, and T has *dp-rank equal to* n if it admits an inp-pattern of depth n but does not admit an inp-pattern of depth $n + 1$. T is said to be *dp-minimal* if it has dp-rank equal to 1.

We can also assume that in the array in Definition 1.1,

(1) The subindices j range over all of $\mathbb{Q}$, which follows by a straightforward compactness argument; and

(2) the array of parameters $\overline{a}_{ij}$ is "mutually indiscernible", that is, for each $i < \kappa$, the sequence $\{\overline{a}_{i,j} : j \in \mathbb{Q}\}$ is indiscernible over the set consisting of the union of all the tuples $\overline{a}_{i',j}$ such that $i' \neq i$ (a proof of this is sketched in [1, Proposition 6]).

Our definition of dp-rank in terms of inp-patterns only applies to NIP theories; in a general theory, inp-patterns correspond to a different rank known as *burden*. For a detailed discussion of these concepts and definitions, see [1] or [10].

Throughout G is an ordered Abelian group in the language $\mathcal{L}_{oag} = \{+, <, 0\}$. Recall that a prime $p \in \mathbb{N}$ is called *singular* for G if $[G : pG] = \infty$.

We state our main result bounding the dp-rank of ordered Abelian groups using some notation which will be defined in the following section.

Theorem 1.2. *Suppose that G is an ordered Abelian group considered as an $\mathcal{L}_{oag}$-structure. Then the dp-rank of G is finite if and only if* **both** *of the following two conditions hold:*

(1) G has only finitely many singular primes; and

(2) for every singular prime p, the auxiliary sort $\mathcal{S}_p$ (see below) is finite.

Moreover, the condition that G is strongly dependent is also equivalent to the conjunction of conditions (1) and (2) above. Furthermore, when these conditions hold, the dp-rank of G is bounded above by

$$1 + \sum_{p \in \mathbb{P}_{sing}} |\mathcal{S}_p|,$$

where $\mathbb{P}_{sing}$ is the set of all primes p which are singular for G.

Theorem 1.2 was established independently by Halevi and Hasson in their preprint "Strongly dependent ordered abelian groups and Henselian fields" [7], as well as by Rafel Farré in the preprint "Strong ordered Abelian groups and dp-rank" [5]. We also note that this characterization of strongly dependent ordered Abelian groups has already been used by Halevi and Hasson to prove that for any strongly dependent pure field K and any henselian valuation v on K, the two-sorted structure (K, vK) is strongly dependent [7].

Our intention in publishing these notes was not to pre-empt either of these works but rather to provide an alternate and potentially more naïve proof of the basic theorem. We encourage the reader to consult either [5] or [7] for more definitive accounts of these results.

2. The languages $\mathcal{L}_{eq}$ and $\mathcal{L}_2$ for quantifier elimination

In this section we will review some notation and fundamental results by Cluckers and Halupczok from [3] on a useful language $\mathcal{L}_{eq}$ for partially eliminating quantifiers in ordered Abelian groups (modulo some quantifiers over linearly ordered auxiliary sorts). We will recall the language $\mathcal{L}_{eq}$, and then we will define a simpler language $\mathcal{L}_2$ (essentially like the langage $\mathcal{L}_{short}$ given by Jahnke, Simon, and Walsberg [8]) which will suffice for eliminating quantifiers when all the $\mathcal{S}_p$ are finite, as is always the case when the group has finite dp-rank (see Theorem 3.1).

The language $\mathcal{L}_{eq}$ of Cluckers and Halupczok is multi-sorted, and to begin we need to recall the definitions of the imaginary sorts $\mathcal{S}_n$, $\mathcal{T}_n$, and $\mathcal{T}_n^+$. As always, G is some ordered Abelian group.

Before continuing, we recall two simple facts for the reader that will be used repeatedly in what follows.

Fact 2.1 Let $(G, +, <)$ be an ordered Abelian group.

(1) G is torsion free.

(2) If $H \subseteq G$ is a convex subroup, $n \in \mathbb{N}$, and $ng \in H$ then $g \in H$.

The following definition is central for the rest of this paper:

Definition 2.1. For a positive integer n and $a \in G \setminus nG$, $H_n(a)$ is the largest convex subgroup of G such that

$$a \notin H_n(a) + nG,$$

which turns out to always be a definable subgroup of G, in fact uniformly definable in a (see [3, Lemma 2.1]). If $a \in nG$, we set $H_n(a) = \{0\}$.

For $a, a' \in G$, say $a \sim a'$ if $H_n(a) = H_n(a')$, and let $\mathcal{S}_n$ be the imaginary sort $G/\sim$. Each of the sorts $\mathcal{S}_n$ is linearly ordered by inclusion. Let $\mathfrak{s}_n : G \to \mathcal{S}_n$ be the canonical surjection.

If $\alpha \in \mathcal{S}_n$ and $\alpha = \mathfrak{s}_n(a)$, then we write "$G_\alpha$" as an abbreviation for $H_n(a)$.

***Remark* 2.2.** As pointed out in [3, Lemma 2.2], for any fixed n, the class of subgroups $\{G_\alpha \,|\, \alpha \in \mathcal{S}_n\}$ is equal to $\{G_\alpha \,|\, \alpha \in \mathcal{S}_p,\, p \text{ is prime, and } p|n\}$. Therefore from now on we will only need to consider sorts $\mathcal{S}_p$ for p prime.

We write $H \leq_{con} G$ if H is a convex subgroup of G.

Definition 2.3. If $\alpha \in \mathcal{S}_p$ and m is a positive integer,

$$G_\alpha^{[m]} = \bigcap \{H + mG \,:\, H \leq_{con} G, H \supsetneq G_\alpha\}.$$

If m, m' are positive integers and $x, y \in G$, write

$$x \equiv_{m,\alpha}^{[m']} y$$

for the relation

$$x - y \in G_\alpha^{[m']} + mG.$$

The groups $G_\alpha^{[m]}$ and the relations $x \equiv_{m,\alpha}^{[m]} y$ are always definable, by the following fact (Lemma 2.4 from [3]):

Fact 2.2.

$$G_\alpha^{[n]} = \bigcap \{G_{\alpha'} + nG \,:\, \alpha' \in \mathcal{S}_n,\, \alpha' > \alpha\}.$$

Given an ordered Abelian group $(G, +, <)$, Cluckers and Halupczok introduce additional auxiliary sorts $\mathcal{T}_p$ and $\mathcal{T}_p^+$ as follows. For any p prime and $b \in G$, let

$$H'_b = \bigcup_{\alpha \in \mathcal{S}_p, b \notin G_\alpha} G_\alpha,$$

and let $\mathcal{T}_p = G/\sim$ where $b \sim b'$ if $H'_b = H'_{b'}$. For $\alpha \in \mathcal{T}_p$, let G_α be the corresponding subgroup (that is, H'_b for any b in the class α). Finally, if $\beta \in \mathcal{T}_p$, let

$$G_{\beta^+} = \bigcap_{\alpha \in \mathcal{S}_p, G_\alpha \supsetneq G_\beta} G_\alpha$$

and define $\mathcal{T}_p^+$ be the corresponding imaginary sort. More precisely, each element of $\mathcal{T}_p^+$ is of the form β^+ for $\beta \in \mathcal{T}_p$, and so $\mathcal{T}_p^+$ is an identical copy of $\mathcal{T}_p$.

We recall the multi-sorted language $\mathcal{L}_{qe}$ which Cluckers and Halupczok used for quantifier elmination in [3].

Definition 2.4. $\mathcal{L}_{qe}$ is the multi-sorted language consisting of a home sort G, infinitely many auxiliary sorts $\{\mathcal{S}_p, \mathcal{T}_p, \mathcal{T}_p^+ : p \text{ is prime}\}$, and the following nonlogical symbols:

(1) A constant symbol for 0, a unary function symbol for $-$, and a binary operation $+$ in the home sort G.

(2) For each pair of primes (p, p'), a binary relation $\leq$ between $(\mathcal{S}_p \cup \mathcal{T}_p \cup \mathcal{T}_p^+)$ and $(\mathcal{S}_{p'} \cup \mathcal{T}_{p'} \cup \mathcal{T}_{p'}^+)$, where "$\alpha \leq \beta$" signifies $G_\alpha \subseteq G_\beta$.

(3) For each prime p and each $\alpha \in \mathcal{S}_p \cup \mathcal{T}_p \cup \mathcal{T}_p^+$, let $\pi : G \to G/G_\alpha$ be the canonical projection map. For each symbol $\diamond \in \{=, <, \equiv_m : m \in \mathbb{N}\}$, the language $\mathcal{L}_{qe}$ contains a ternary relation $\diamond_\alpha$ on $G \times G \times (\mathcal{S}_p \cup \mathcal{T}_p \cup \mathcal{T}_p^+)$ such that $x \diamond_\alpha y$ holds if and only if $\pi(x) \diamond \pi(y)$ holds in G/G_α (where "$\equiv_m$" denotes the relation of congruence modulo m within G/G_α).

(4) With notation as in the previous point, for any $k \in \mathbb{Z}$, let k_α be k times the minimal positive element of the group G/G_α in case this group is discrete, and otherwise let k_α be the zero element of this quotient. Then for each $k \in \mathbb{Z}$ and each $\diamond \in \{=, <, \equiv_m : m \in \mathbb{N}\}$, there are symbols for ternary relations on $G \times G \times (\mathcal{S}_p \cup \mathcal{T}_p \cup \mathcal{T}_p^+)$ denoting the relation $\pi(x) \diamond \pi(y) + k_\alpha$ on the triple (x, y, α).

(5) For each $m, m' \in \mathbb{N}$ and p prime, there is a ternary relation symbol on $G \times G \times \mathcal{S}_p$ for the relation $x \equiv_{m,\alpha}^{[m']}$ defined above.

(6) For each prime p, a unary predicate $\operatorname{discr}(\alpha)$ on $\mathcal{S}_p$ which holds if and only if G/G_α is discretely ordered.

(7) For each prime p, each $s \in \mathbb{N} \setminus \{0\}$ and each $\ell \in \mathbb{N}$, two more unary predicates on $\mathcal{S}_p$ defining the sets

$$\{\alpha \in \mathcal{S}_p \,:\, \dim_{\mathbb{F}_p}(G_\alpha^{[p^s]} + pG)/(G_\alpha^{[p^{s+1}]} + pG) = \ell\}$$

and

$$\{\alpha \in \mathcal{S}_p \,:\, \dim_{\mathbb{F}_p}(G_\alpha^{[p^s]} + pG)/(G_\alpha + pG) = \ell\}.$$

Any ordered Abelian group can naturally be interpreted as an $\mathcal{L}_{eq}$-structure as indicated above.

Cluckers and Halupczuk prove the following result on partial quantifier elimination:

Fact 2.3. ([3], Theorem 1.8) Suppose that $\varphi(\overline{x}, \overline{\eta})$ is any $\mathcal{L}_{qe}$-formula, where $\overline{x}$ are the variables from the home sort and $\overline{\eta}$ are the variables from the sorts $\mathcal{S}_p, \mathcal{T}_p$ and $\mathcal{T}_p^+$. Then in the theory of ordered Abelian groups, $\varphi(\overline{x}, \overline{\eta})$ is equivalent to an $\mathcal{L}_{qe}$-formula of the form

$$\bigvee_{i=1}^{k} \exists \overline{\theta} \left(\xi_i(\overline{\eta}, \overline{\theta}) \wedge \psi_i(\overline{x}, \overline{\theta}) \right),$$

where $\overline{\theta}$ are variables from the sorts $\mathcal{S}_p, \mathcal{T}_p$ and $\mathcal{T}_p^+$ and the formulas $\psi_i(\overline{x}, \overline{\theta})$ are conjunctions of literals (atomic formulas or their negations).

The following fact is a direct consequence of the quantifier reduction result above, and is proved by a similar argument as in [8].

Corollary 2.5. *Suppose that for every prime p, the sort $\mathcal{S}_p$ is finite. Then the complete theory of $(G, <, +, 0)$ eliminates quantifiers in the (single-sorted) extension $\mathcal{L}_2$ of $\mathcal{L}_{oag}$ which contains the following additional symbols:*

(1) A unary function symbol for $-$.

(2) Binary predicates $\equiv_m$ for the relation $x - y \in mG$, for each positive $m \in \mathbb{N}$.

(3) Binary predicates $\equiv_{m,\alpha}$ for each positive $m \in \mathbb{N}$, $\alpha \in \mathcal{S}_p$, and p a prime denoting the relation

$$x \equiv_{m,\alpha} y \Leftrightarrow x - y \in G_\alpha + mG;$$

(4) Unary predicates for the (countably many) convex subgroups G_α, where $\alpha \in \mathcal{S}_p$ for some prime p.

(5) Constants naming a countable elementary submodel G_0 of G.

Proof. First note that since $S_p := \{G_\alpha : \alpha \in \mathcal{S}_p\}$ is a finite set of groups linearly ordered by inclusion, and each group G_α named by $\alpha \in \mathcal{T}_p \cup \mathcal{T}_p^+$ is a union or intersection of groups in S_p, any such G_α is in fact in S_p (or else is $\{0\}$ or all of G). Therefore we need only consider the sorts $\mathcal{S}_p$ and the home sort G.

Suppose that $\varphi(\overline{x})$ is any $\mathcal{L}_{oag}$-formula. By Fact 2 above, it is equivalent (modulo the theory of ordered Abelian groups) to an $\mathcal{L}_{qe}$-formula of the form

$$\bigvee_{i=1}^{k} \exists \overline{\theta} \left(\xi_i(\overline{\theta}) \wedge \psi_i(\overline{x}, \overline{\theta}) \right)$$

where the $\psi_i(\overline{x}, \overline{\theta})$ are conjunctions of literals in $\mathcal{L}_{qe}$. So assume that $\varphi(\overline{x})$ is in this form, with the variables $\overline{\theta}$ coming from sorts $\mathcal{S}_p$, as explained above.

Observe that each variable from $\overline{\theta}$ ranges over a single sort $\mathcal{S}_p$, which is assumed to be finite, so that the existential quantifiers are equivalent to finite disjunctions over all of their possible values. So without loss of generality, we can assume that φ is of the form

$$\bigvee_{i=1}^{k} \psi_i(\overline{x}, \overline{\theta}_i)$$

where the ψ_i are conjunctions of literals as before and $\overline{\theta_i}$ are fixed values for parameters in the sorts $\mathcal{S}_p$.

Now we can go through each of the nonlogical symbols of $\mathcal{L}_{qe}$ (from parts (2) through (7) of Definition 2.4) and verify that they can be replaced by symbols from $\mathcal{L}_2$ in φ:

- Clause (2) gives binary symbols $\leq$ in $\mathcal{L}_{qe}$ between the auxiliary sorts $\mathcal{S}_p$ and $\mathcal{S}_{p'}$. Such symbols would only affect the truth conditions on atomic subformulas $\psi(\overline{\theta}_i)$ of $\psi_i(\overline{x}, \overline{\theta}_i)$ which do not involve the variables $\overline{x}$, since there are no functions from the sort G into the auxiliary sorts $\mathcal{S}_p$, and thus any such $\psi(\overline{\theta}_i)$ can simply be replaced by $x = x$ or $\neg(x = x)$.

- Clause (3) defines ternary relations on $G \times G \times \mathcal{S}_p$ corresponding to the relations of $=, <$, and $\equiv_m$ on quotient groups G/G_α. But since $\mathcal{S}_p$ is finite and each of its corresponding subgroups is named by a predicate in $\mathcal{L}_2$, the relation $G_{\pi(x)} = G_{\pi(y)}$ is expressible as a finite disjunction of atomic $\mathcal{L}_2$-formulas in x and y, and the containment relation $G_{\pi(x)} < G_{\pi(y)}$ is similarly definable. Finally, for the relation $G_{\pi(x)} \equiv_m G_{\pi(y)}$, we note that this is equivalent to the relation $x \equiv_{m,\alpha} y$ which we have included in $\mathcal{L}_2$. (*Proof:* this translates into showing that

  $$x - y \in G_\alpha + mG \Leftrightarrow (x - y) + G_\alpha \in m(G/G_\alpha).$$

  But if $x - y = g + mh$ with $z \in G_\alpha$ and $h \in G$, then $x - y \in m(h + G_\alpha) \in m(G/G_\alpha)$; and conversely if $(x-y)+G_\alpha = mh+G_\alpha$, then $x-y = mh+g$ for some $g \in G_\alpha$, and thus $x - y \in G_\alpha + mG$.)

- Clause (4) gives symbols for relations $\pi(x) \diamond_\alpha \pi(y) + k_\alpha$. But if G/G_α is discrete, then for any $k \in \mathbb{Z}$ there must be $a \in G_0$ such that $\pi(z) = k_\alpha$, and so with a constant symbol from $\mathcal{L}_2$ for such an element a we can define these relations as above.

- Clause (5) gives symbols for the relations $\equiv_{m,\alpha}^{[m']}$, but by Fact 2, when every $\mathcal{S}_p$ is finite these are equivalent to instances of $\equiv_{m,\alpha}$.

- The predicates discr(α) on $\mathcal{S}_p$ (Clause (6)) can be eliminated in the same way as the relations $\leq$ from Clause (2), and likewise for the other unary predicates on $\mathcal{S}_p$ given in Clause (7).

☑

3. Infinite $\mathcal{S}_p$ implies not strongly dependent

In [8] Jahnke, Simon, and Walsberg show that if G has no non-singular primes then G is dp-minimal. In [2] Chernikov, Kaplan, and Simon show that if G has infinitely many singular primes then G is not strongly dependent. Thus it is obvious to conjecture that if G has only finitely many singular primes then G is strongly dependent (and in fact of finite dp-rank). In this section we show that this is false and that in fact a group can have only one singular prime and nonetheless still not be strongly dependent.

For convenience, notice the following facts, which are inherent in [3].

Fact 3.1. If $a, b \in G$ are equivalent modulo pG then $H_p(a) = H_p(b)$. In particular if p is non-singular then $\mathcal{S}_p$ is finite.

Proof. This follows immediately from the definitions, since for a and b which are equivalent modulo pG and any convex subgroup H of G, we have that $a \in H + pG$ if and only if $b \in H + pG$. ☑

Fact 3.2. If $H_p(a) \subset H_p(b)$ then we can find $a' \in H_p(b)$ so that $H_p(a) = H_p(a')$.

Proof. Note that by definition $a \in H_p(b) + pG$. Let $a = a' + pg$ where $a' \in H_p(b)$. Then a and a' are equivalent modulo pG and thus by the previous fact $H_p(a) = H_p(a')$. ☑

Theorem 3.1. *Suppose that for some prime $\mathcal{S}_p$ is infinite. Then G is not strongly dependent.*

Proof. Without loss of generality we may assume that G is an ω_1-saturated model of $\mathrm{Th}(G)$. Thus we may find elements $e_i, f_{i,j} \in G$ for $i, j \in \omega$ such that

$$H_p(e_0) \subset H_p(e_1) \subset \dots$$

and for each $i, j \in \omega$,

$$H_p(e_i) \subset H_p(f_{i,j}) \subset H_p(f_{i,j+1}) \subset H_p(e_{i+1}).$$

By Fact 3 we may assume that $e_i \in H_p(e_{i+1})$ and $f_{i,j} \in H_p(f_{i,j+1})$.

Let $c_{i,j} = p^i f_{i,j}$ and let α_i be the element of the sort $\mathcal{S}_p$ such that $G_{\alpha_i} = H_p(e_i)$.

Claim. If $j_0 \neq j_1$, then

$$c_{i,j_0} \not\equiv_{p^{i+1},\alpha_i} c_{i,j_1}.$$

Proof. Without loss of generality, $j_0 < j_1$. Suppose, towards a contradiction, that these two elements are $\equiv_{p^{i+1},\alpha_i}$-equivalent. Then there are $g \in H_p(e_i)$ and $h \in G$ such that

$$c_{i,j_0} - c_{i,j_1} = p^i f_{i,j_0} - p^i f_{i,j_1} = g + p^{i+1} h.$$

Thus the element g is divisible by p^i, and by convexity of $H_p(e_i)$ there is $g' \in H_p(e_i)$ such that $g = p^i g'$.

Now comparing with the previous displayed equation above, we can cancel out the factors of p^i to conclude that

$$f_{i,j_0} - f_{i,j_1} = g' + ph,$$

so

$$f_{i,j_1} = -g' - ph + f_{i,j_0} \in pG + H_p(f_{i,j_1})$$

since $j_0 < j_1$ implies that $f_{i,j_0} \in H_p(f_{i,j_1})$ and $g' \in H_p(e_i) \subseteq H_p(f_{i,j_1})$. But this contradicts the definition of $H_p(f_{i,j_1})$. ☑

Claim. For any $n \in \omega$ and any $\eta : [n] \to [n]$ (where $[n] = \{1, 2, \ldots, n\}$), the formula

$$\bigwedge_{i=1}^{n} x \equiv_{p^{i+1},\alpha_i} c_{i,\eta(i)}$$

is consistent, and is satisfied by the element

$$a := \sum_{i=1}^{n} c_{i,\eta(i)}.$$

Proof. It suffices to show that if $i \in [n]$ and $j \in [n] \setminus \{i\}$, then $c_{j,\eta(j)} \in p^{i+1}G + H_p(e_i)$. But on the one hand, if $j < i$, then

$$c_{j,\eta(j)} = p^j f_{j,\eta(j)} \in H_p(f_{j,\eta(j)+1}) \subseteq H_p(e_i)$$

and, on the other hand, if $j > i$, then $c_{j,\eta(j)} = p^j f_{j,\eta(j)} \in p^{i+1}G$. ☑

By the preceding two claims, we conclude that there is an inp-pattern of depth ω in G whose i-th row consists of the formulas of the form

$$x \equiv_{p^{i+1},\alpha_i} c_{i,j}$$

as j varies over ω. ☑

Notice that Theorem 3.1 together with the result from [2] mentioned above established the "only if" portion of Theorem 1.2.

4. Bounding the dp-rank of G from above

In this section we will prove that if conditions (1) and (2) of Theorem 1.2 hold, then the dp-rank of G is finite. At the same time, we will also establish the upper bound on the dp-rank given in Theorem 1.2.

From now on, we work in an ordered Abelian group G such that the sorts $\mathcal{S}_p$ are all finite, and we fix such an inp-pattern of depth n in the language $\mathcal{L}_2$ described above. By quantifier elimination, we may assume that each formula φ_i is quantifier-free.

The following is easy and already known, but we include it for convenient reference:

Lemma 4.1. *We may further assume that each formula φ_i is a conjunction of literals (atomic formulas or negations of atomic formulas).*

***Proof*.** Write each φ_i as a disjunction of conjunctions of literals, say

$$\varphi_i(x;\overline{y}_i) = \bigvee_{\ell=1}^{m_i} \theta_{i,\ell}(x;\overline{y}_j).$$

Then there are $\ell_1, \dots, \ell_n$ such that

$$\theta_{1,\ell_1}(x;\overline{a}_{1,0}) \wedge \ldots \wedge \theta_{n,\ell_n}(x;\overline{a}_{n,0})$$

is consistent. Replace each formula φ_i by θ_{i,ℓ_i}. The k_i-inconsistency of each row is clearly preserved, and the mutual indiscernibility of the parameters ensures that we also have the consistency condition we require. ☑

Now we need to consider in more detail the literals which constitute each formula $\varphi_i(x;\overline{y}_i)$.

Note that we may safely assume that each literal in every formula $\varphi_i(x;\overline{y}_i)$ actually mentions the variable x. Furthermore, we may use the fact that $\mathcal{L}_2$-terms are linear functions of their variables to put every literal in every formula $\varphi_i(x;\overline{y}_i)$ into one of the following four types (we allow the parameter α to name the subgroup $\{0\}$):

Type (I): $kx \equiv_{m,\alpha} t(\overline{y}_i)$ for some $k, m \in \mathbb{N} \setminus \{0\}$, $\alpha \in \mathcal{S}_p$ with p a prime, and $\mathcal{L}_{oag}$-term $t(\overline{y}_i)$.

Type (II): $\neg(kx \equiv_{m,\alpha} t(\overline{y}_i)$ for some k, m, α, p, and $t(\overline{y}_i)$ as above.

Type (III): Literals of the form $kx \diamond t(\overline{y}_i)$ where $\diamond \in \{<, >, \leq, \geq, =\}$, or of the type $kx \in G_\alpha + t(\overline{y}_i)$, where t is a term and $\alpha \in \mathcal{S}_p$ for some prime p.

Type (IV): Literals of the form $kx \neq t(\overline{y}_i)$ or $kx \notin G_\alpha + t(\overline{y}_i)$, with k, t, and α as above.

Since we allow α to name $\{0\}$, the division into the four types is exhaustive as the literals of Type (I) and (II) encompass the simple congruence relations $x \equiv_m t(\overline{y}_i)$ and $\neg(x \equiv_m t(\overline{y}_i))$.

Assumption 4.2. For each $i \in \{1, \dots, n\}$, the formula $\varphi_i(x; \overline{y}_i)$ is a *minimal* conjunction of literals, in the sense that if we were to remove any one of these literals from the conjunction, then the resulting row of formulas

$$\{\varphi_i'(x; \overline{a}_{i,j}) : j \in \mathbb{Q}\}$$

would be consistent.

Proposition 4.2. *Under Assumption 4, each formula $\varphi_i(x, \overline{y}_i)$ is either (a) a* ***single*** *formula $kx \equiv_{m,\alpha} t(\overline{y})$ of Type (I), or else (b) a conjunction of literals of Type (III).*

Proof. Call a literal $\psi(x; \overline{a}_{ij})$ occurring in $\varphi_i(x; \overline{a}_{ij})$ *fixed* if it defines the same subset of G even as j varies, and call it *variable* otherwise.

As a first reduction, suppose that $\varphi_i(x; \overline{a}_{ij})$ contains a variable literal of Type (I). Note that literals of Type (I) define cosets of subgroups of G, so by Assumption 4 this Type (I) literal must be the only conjunct in $\varphi_i(x; \overline{a}_{ij})$, and we are done. Thus we may assume that any literal in $\varphi_i(x; \overline{a}_{ij})$ of Type (I) is fixed, and our goal will be to show that in fact every literal is of Type (III).

Write

$$\varphi_i(x; \overline{y}_i) = \psi_1(x; \overline{y}_i) \wedge \psi_2(x; \overline{y}_i) \wedge \psi_3(x; \overline{y}_i),$$

where:

- ψ_1 is the conjunction of all fixed literals of Type (I), (II) or (IV),
- ψ_2 is the conjunction of all variable literals of Type (II) or (IV), and
- ψ_3 is the conjunction of all Type (III) literals.

Note that the sets defined by the $\psi_3(x, \overline{a}_{i,j})$ are convex.

We allow the possibility that there are no literals of one of these types, in which case the corresponding $\psi_i(x; \overline{a}_{ij})$ is equivalent to $x = x$. We will assume, towards a contradiction, that not all literals of $\varphi_i(x; \overline{a}_{ij})$ are contained in $\psi_3(x; \overline{a}_{ij})$. Thus by minimality, $\{\psi_3(x; \overline{a}_{ij}) : j \in \mathbb{Q}\}$ is consistent.

Claim. For any $j \in \mathbb{Q}$, there is a finite $F \subseteq \mathbb{Q} \setminus \{j\}$ such that the formula

$$\psi_1(x; \overline{a}_{ij}) \wedge \psi_3(x; \overline{a}_{ij}) \wedge \bigwedge_{j' \in F} \psi_2(x; \overline{a}_{ij'})$$

is inconsistent.

Proof. Recall that row i is k_i-inconsistent.

Case 1: The convex sets defined by the $\psi_3(x; \overline{a}_{ij})$ are nested: that is, there are distinct $\alpha, \beta \in \omega$ such that $\psi_3(G; \overline{a}_{i\alpha}) \subseteq \psi_3(G; \overline{a}_{i\beta})$.

In this case, if $j \in \mathbb{Q}$, by indiscernibility we can pick distinct $\alpha(1), \ldots, \alpha(k_i) \in \mathbb{Q}$ such that

$$\psi_3(G; \overline{a}_{i\alpha(1)}) \supseteq \ldots \supseteq \psi_3(G; \overline{a}_{i\alpha(k_i)}) \supseteq \psi_3(G; \overline{a}_{ij})$$

and, by k_i-inconsistency of the ith row of the inp-pattern,

$$\psi_1(x; \overline{a}_{ij}) \wedge \psi_3(x; \overline{a}_{ij}) \wedge \bigwedge_{\ell=1}^{k_i} \psi_2(x; \overline{a}_{i\alpha(\ell)})$$

is inconsistent (recall that ψ_1 is fixed).

Case 2: The convex sets defined by the $\psi_3(x; \overline{a}_{ij})$ are not nested as j varies.

In this case, fix $j < j'$ and pick an element $c_1 \in \psi_3(G; \overline{a}_{ij}) \setminus \psi_3(G; \overline{a}_{ij'})$. By convexity of $\psi_3(G; \overline{a}_{ij'})$, either $c_1 < \psi_3(G; \overline{a}_{ij'})$ (meaning that c_1 is less than every element of $\psi_3(G; \overline{a}_{ij'})$) or else $\psi_3(G; \overline{a}_{ij'}) < c_1$. Without loss of generality, assume that we are in the former case; then for *any* $c \in \psi(G; \overline{a}_{ij})$, either $c < \psi_3(G; \overline{a}_{ij'})$ or $c \in \psi_3(G; \overline{a}_{ij'})$, since otherwise $c > \psi_3(G; \overline{a}_{ij'})$ and the convexity of $\psi_3(G; \overline{a}_{ij})$ would imply that $\psi_3(G; \overline{a}_{ij'}) \subseteq \psi_3(G; \overline{a}_{ij})$, contradicting the fact that we are in Case 2.

By the preceding observation and indiscernibility, whenever $j < j' < j''$,

$$\psi_3(G; \overline{a}_{ij}) \cap \psi_3(G; \overline{a}_{ij''}) \subseteq \psi_3(G; \overline{a}_{ij'}) \cap \psi_3(G; \overline{a}_{ij''}), \tag{1}$$

since for any $c \in \psi_3(G; \overline{a}_{ij}) \cap \psi_3(G; \overline{a}_{ij''})$ either $c \in \psi_3(G; \overline{a}_{ij'})$ or else $c < \psi_3(G; \overline{a}_{ij'})$, but $c < \psi_3(G; \overline{a}_{ij'})$ would imply that $c < \psi_3(G; \overline{a}_{ij''})$ (since there is an element $c' \in \psi_3(G; \overline{a}_{ij'})$ such that $c' < \psi_3(G; \overline{a}_{ij''})$), which gives a contradiction.

Furthermore, if $j < j' < j''$, then by the consistency of $\{\psi_3(x; \overline{a}_{i,j}) : j \in \mathbb{Q}\}$, indiscernibility, and the convexity of the sets each of these formulas define,

$$\psi_3(G; \overline{a}_{ij'}) \subseteq \psi_3(G; \overline{a}_{ij}) \cup \psi_3(G; \overline{a}_{ij''}). \tag{2}$$

Given any $j \in \mathbb{Q}$, choose elements $j(1) < \ldots < j(k_i) < j < j(k_i+1) < \ldots < j(2k_i)$. Then

$$\psi_3(G; \overline{a}_{ij}) \subseteq \left(\bigcap_{\ell=1}^{k_i} \psi_3(G; \overline{a}_{ij(\ell)}) \cup \bigcap_{\ell=k_i+1}^{2k_i} \psi_3(G; \overline{a}_{ij(\ell)}) \right), \tag{3}$$

since, for any $c \in \psi_3(G; \overline{a}_{ij})$, by (2), either $c \in \psi_3(G; \overline{a}_{ij(1)}) \cap \psi_3(G; \overline{a}_{ij})$ or else $c \in \psi_3(G; \overline{a}_{ij}) \cap \psi_3(G; \overline{a}_{ij(2k_i)})$ and, in the former case, applying (1) yields that $c \in \bigcap_{\ell=1}^{k_i} \psi_3(G; \overline{a}_{ij(\ell)})$, while in the latter case, applying (1) gives us that $c \in \bigcap_{\ell=k_i+1}^{2k_i} \psi_3(G; \overline{a}_{ij(\ell)})$.

By the k_i-inconsistency of row i, both

$$\psi_1(x; \overline{a}_{ij}) \wedge \bigwedge_{\ell=1}^{k_i} \psi_3(x; \overline{a}_{ij(\ell)}) \wedge \bigwedge_{\ell=1}^{k_i} \psi_2(x; \overline{a}_{ij(\ell)})$$

and

$$\psi_1(x; \overline{a}_{ij}) \wedge \bigwedge_{\ell=k_i+1}^{2k_i} \psi_3(x; \overline{a}_{ij(\ell)}) \wedge \bigwedge_{\ell=k_i+1}^{2k_i} \psi_2(x; \overline{a}_{ij(\ell)})$$

are inconsistent. So by formula (3) above,

$$\psi_1(x; \overline{a}_{ij}) \wedge \psi_3(x; \overline{a}_{ij}) \wedge \bigwedge_{\ell=1}^{2k_i} \psi_2(x; \overline{a}_{ij(\ell)})$$

is inconsistent, as desired. ☑

Note that the Claim just proved implies that there must be at least one literal occurring in the conjunction $\psi_2(x; \overline{y})$ (as we achieve inconsistency by only varying the parameters in ψ_2).

Now suppose that $\neg\theta(x; \overline{y})$ is any literal of Type (II) or (IV) occurring in $\psi_2(x; \overline{y})$, where $\theta(x; \overline{y})$ is an atomic formula. We will establish the following claim, which will contradict our minimality assumption on $\varphi_i(x; \overline{y}_i)$ and finish the proof of Proposition 4.2:

Claim. If $\widehat{\psi_2}(x; \overline{y})$ is the smaller conjunction obtained by removing $\neg\theta(x; \overline{y})$ from $\psi_2(x; \overline{y})$, then

$$\{\psi_1(x; \overline{a}_{ij}) \wedge \widehat{\psi_2}(x; \overline{a}_{ij}) \wedge \psi_3(x; \overline{a}_{ij}) : j \in \mathbb{Q}\}$$

is inconsistent.

Proof. Fix any $j \in \mathbb{Q}$. By the previous Claim, there is a finite $F \subseteq \mathbb{Q} \setminus \{j\}$ such that

$$\psi_1(x; \overline{a}_{ij}) \wedge \psi_3(x; \overline{a}_{ij}) \wedge \bigwedge_{j' \in F} \psi_2(x; \overline{a}_{ij'})$$

is inconsistent. Now by the fact that the literals in $\psi_2(x; \overline{a}_{ij})$ are variables of Type (II) or (IV) and thus define cosets, it follows that if $j' \neq j$, then $\theta(x; \overline{a}_{ij})$ implies $\neg\theta(x; \overline{a}_{ij'})$, and hence

$$\psi_1(x; \overline{a}_{ij}) \wedge \psi_3(x; \overline{a}_{ij}) \wedge \theta(x; \overline{a}_{ij}) \wedge \bigwedge_{j' \in F} \widehat{\psi_2}(x; \overline{a}_{ij'})$$

is also inconsistent. This means that the formula

$$\varphi_i(x;\overline{a}_{ij}) = \psi_1(x;\overline{a}_{ij}) \wedge \psi_2(x;\overline{a}_{ij}) \wedge \psi_3(x;\overline{a}_{ij})$$

is implied by

$$\psi_1(x;\overline{a}_{ij}) \wedge \psi_3(x;\overline{a}_{ij}) \wedge \widehat{\psi_2}(x;\overline{a}_{ij}) \wedge \bigwedge_{j'\in F} \widehat{\psi_2}(x;\overline{a}_{ij'}). \tag{4}$$

Now pick any family of pairwise disjoint subsets $F_1, \dots, F_{k_i}$ of $\mathbb{Q} \setminus \{1, \dots, k_i\}$ such that $|F_\ell| = |F|$ for every ℓ and let $F' = F_1 \cup \ldots \cup F_{k_i} \cup \{1, \dots, k_i\}$ and consider

$$\bigwedge_{j\in F'} \psi_1(x;\overline{a}_{ij}) \wedge \widehat{\psi_2}(x;\overline{a}_{ij}) \wedge \psi_3(x;\overline{a}_{ij}). \tag{5}$$

If we let

$$\xi_j(x) = \psi_1(x;\overline{a}_{ij}) \wedge \psi_3(x;\overline{a}_{ij}) \wedge \widehat{\psi_2}(x;\overline{a}_{ij}) \wedge \bigwedge_{j'\in F_j} \widehat{\psi_2}(x;\overline{a}_{ij'})$$

for $j \in \{1, \dots, k_i\}$, then the formula (5) implies the formula $\xi_1(x) \wedge \ldots \wedge \xi_{k_i}$. By previous remarks and indiscernibility, this in turn implies

$$\bigwedge_{j=1}^{k_i} \varphi_i(x;\overline{a}_{ij})$$

which is inconsistent. Thus we have a finite inconsistent conjunction of formulas of the form $\psi_1(x;\overline{a}_{ij}) \wedge \widehat{\psi_2}(x;\overline{a}_{ij}) \wedge \psi_3(x;\overline{a}_{ij})$ as desired. ☑

Thus the previous claim shows that if a literal of the form $\neg\theta(x,\overline{y})$ of either Type (II) or (IV) occurs in $\psi_2(x,\overline{y}_i)$ it can be deleted from $\psi_2(x,\overline{y}_i)$ and inconsistency is preserved. This violates the minimality of $\varphi_i(x,\overline{y}_i)$ and hence no such literal occurs.

This finishes the proof of Proposition 4.2. ☑

Lemma 4.3. *There is at most one $i \in \{1, \dots, n\}$ such that row i consists of a conjunction of literals of Type (III).*

Proof. Since literals of Type (III) define convex sets, this follows by the same argument as was used in Theorem 4.1 of [4]; for clarity, we reproduce the argument here.

Suppose that the formula $\varphi_i(x;\overline{y}_i)$ in Row i is a conjunction of literals of Type (III). Each instance of a literal of Type (III) defines a convex set, so each formula $\varphi_i(x;\overline{a}_{ij})$ defines a convex set. The intersection of a family $\mathcal{F}$ of convex sets in a linearly ordered structure is nonempty if and only if the intersection of any two sets from $\mathcal{F}$ is nonempty, so by indiscernibility and k_i-inconsistency, the set $\{\varphi_i(x;\overline{a}_{ij})\}$ is 2-inconsistent.

Now assume, towards a contradiction, that both the formulas $\varphi_{i_1}(x;\overline{y}_{i_1})$ in Row i_1 and the formulas $\varphi_{i_2}(x;\overline{y}_{i_2})$ in Row i_2 are conjunctions of literals of Type (III). By the preceding paragraph, both rows are 2-inconsistent, and therefore define bounded, pairwise disjoint convex subsets of G. Suppose that $j_1, j_2 \in \mathbb{Q}$ are chosen such that $\varphi_{i_1}(G;\overline{a}_{i_1,j_1}) < \varphi_{i_1}(G;\overline{a}_{i_1,j_2})$, meaning that each element of the first set precedes every element of the second in the linear ordering, and likewise $j_1', j_2' \in \mathbb{Q}$ are such that $\varphi_{i_2}(G;\overline{a}_{i_2,j_1'}) < \varphi_{i_2}(G;\overline{a}_{i_2,j_2'})$. Now $\varphi_{i_1}(G;\overline{a}_{i_1,j_2}) \cap \varphi_{i_2}(G;\overline{a}_{i_2,j_1'})$ contains some element b (by the definition of an inp-pattern), and therefore our assumptions imply that

$$\varphi_{i_1}(G;\overline{a}_{i_1,j_1}) < b < \varphi_{i_2}(G;\overline{a}_{i_2,j_2'}).$$

But this contradicts the requirement that $\varphi_{i_1}(G;\overline{a}_{i_1,j_1}) \cap \varphi_{i_2}(G;\overline{a}_{i_2,j_2'})$ be nonempty since it is the intersection of sets from two different rows of the inp-pattern. ☑

Next, we further simplify the literals of Type (I) which appear as rows in our inp-pattern.

Lemma 4.4. *Without loss of generality, each formula $\varphi_i(x;\overline{y})$ of Type (I) which appears in the inp-pattern is of the form*

$$x \equiv_{p^\ell,\alpha} t(\overline{y})$$

for some singular prime p, $\ell \in \mathbb{N}$, and some $\mathcal{L}_{oag}$-term $t(\overline{y})$.

Proof. Suppose the formula in the i-th row is $kx \equiv_{m,\alpha} t(\overline{y}_i)$.

Claim. Without loss of generality, $m = p^\ell$ for some singular prime p.

Proof. Note that if $m = m_1 m_2$ with m_1, m_2 relatively prime, then

$$kx \equiv_{m,\alpha} t(\overline{a}_{ij}) \Leftrightarrow (kx \equiv_{m_1,\alpha} t(\overline{a}_{ij}) \wedge kx \equiv_{m_2,\alpha} t(\overline{a}_{ij}))\,.$$

(This statement is simply a version of the Chinese remainder theorem; see Lemma 2.7 of [3].) Thus $kx \equiv_{m,\alpha} t(\overline{a}_{ij})$ is equivalent to a conjunction of congruences of the form $kx \equiv_{p^\ell,\alpha}$ for prime powers p^ℓ. For the ith row to be inconsistent, there must be some such prime power p^ℓ such that $\{kx \equiv_{p^\ell,\alpha} t(\overline{a}_{ij}) : j \in \mathbb{Q}\}$ is inconsistent; then it is clear that p must be a singular prime, and that we may replace the ith row with these formulas. ☑

Claim. Without loss of generality, p does not divide k.

Proof. Suppose that $p|k$. Since each formula in the inp-pattern is consistent, $t(\overline{a}_{ij}) \in pG + G_\alpha$ for every $j \in \mathbb{Q}$. Also, we may extend the tuples $\overline{a}_{ij}$ in the indiscernible sequence if necessary so that they include elements $a'_{ij} \in \overline{a}_{ij}$ such that $t(\overline{a}_{ij}) \in pa'_{ij} + G_\alpha$. (To preserve the indiscernibility first add the new elements a'_{ij} if necessary then to obtain a new array and then argue once again as in [1, Proposition 6] to obtain on indiscernible array with the same properties.)

Note that p^ℓ cannot divide k since otherwise the formula $kx \equiv_{p^\ell,\alpha} t(\overline{a}_{ij})$ would be either always true or always false (independently of the value of j) and this could not form the row of an inp-pattern. We assert that for any $x \in G$,

$$kx \equiv_{p^\ell,\alpha} t(\overline{a}_{ij}) \Leftrightarrow (k/p)x \equiv_{p^{\ell-1},\alpha} a'_{ij}$$

and the Claim follows by applying this repeatedly until no factors of p in k remain. To see why the assertion is true, suppose on the one hand that $kx \equiv_{p^\ell,\alpha} t(\overline{a}_{ij})$; then $kx \equiv_{p^\ell,\alpha} pa'_{ij}$, and so

$$p\left((k/p)x - a'_{ij}\right) \in G_\alpha + p^\ell G.$$

So we can write

$$p\left((k/p)x - a'_{ij}\right) = g + p^\ell h$$

with $g \in G_\alpha$ and $h \in G$. Then g is p-divisible, and furthermore $g = pg_0$ for some $g_0 \in G_\alpha$ by Fact 2; thus

$$p\left((k/p)x - a'_{ij}\right) = p(g_0 + p^{\ell-1}h)$$

$$\Rightarrow (k/p)x - a'_{ij} = g_0 + p^{\ell-1}h,$$

and so $(k/p)x \equiv_{p^{\ell-1},\alpha} a'_{ij}$ as desired. Conversely, if

$$(k/p)x - a'_{ij} = g + p^{\ell-1}h$$

for $g \in G_\alpha$ and $h \in G$, then multiplying by p gives

$$kx - t(\overline{a}_{ij}) + G_\alpha = p^\ell h + G_\alpha,$$

and so $kx \equiv_{p^\ell,\alpha} t(\overline{a}_{ij})$. ☑

Notice that the preceding claim and its proof only relied on the syntactic form of the formula not on the properties of a specific instance of the formula.

Finally, we have reduced to the case of a formula $kx \equiv_{p^\ell,\alpha} t(\overline{a}_{ij})$ where $\gcd(p^\ell, k) = 1$. Pick integers r, s such that $rp^\ell + sk = 1$. We claim that for any $x \in G$,

$$kx \equiv_{p^\ell,\alpha} t(\overline{a}_{ij}) \Leftrightarrow x \equiv_{p^\ell,\alpha} s \cdot t(\overline{a}_{ij}),$$

completing the proof of Lemma 4.4. On the one hand, if $kx \equiv_{p^\ell,\alpha} t(\overline{a}_{ij})$ and

$$kx - t(\overline{a}_{ij}) = g + p^\ell h$$

with $g \in G_\alpha$ and $h \in G$, then

$$\begin{aligned} skx - s \cdot t(\overline{a}_{ij}) &= sg + p^\ell sh \\ \Rightarrow (1 - rp^\ell)x - s \cdot t(\overline{a}_{ij}) &= sg + p^\ell sh \\ \Rightarrow x - s \cdot t(\overline{a}_{ij}) &= sg + p^\ell (sh + rx), \end{aligned}$$

so $x \equiv_{p^\ell,\alpha} s \cdot t(\overline{a}_{ij})$. On the other hand, if $x \equiv_{p^\ell,\alpha} s \cdot t(\overline{a}_{ij})$ and

$$x - s \cdot t(\overline{a}_{ij}) = g + p^\ell h$$

with $g \in G_\alpha$ and $h \in G$, then

$$\begin{aligned} kx - ks \cdot t(\overline{a}_{ij}) &= kg + p^\ell kh \\ \Rightarrow kx - (1 - rp^\ell) t(\overline{a}_{ij}) &= kg + p^\ell kh \\ \Rightarrow kx - t(\overline{a}_{ij}) &= kg + p^\ell (kh - rt(\overline{a}_{ij})), \end{aligned}$$

so $kx \equiv_{p^\ell,\alpha} t(\overline{a}_{ij})$. ☑

Lemma 4.5. *Suppose that two different rows of the inp-pattern, Row i and Row i′, consist of Type (I) formulas*

$$x \equiv_{p^\ell,\alpha} t(\overline{y}_i)$$

and

$$x \equiv_{p^{\ell'},\alpha'} t'(\overline{y}_{i'})$$

respectively, with the same singular prime p.

Then if $G_\alpha \subseteq G_{\alpha'}$, there is some $\beta \in \mathcal{S}_p$ such that $\alpha \leq \beta < \alpha'$.

Proof. Recall that the convex subgroups of G are linearly ordered by inclusion. First note that $\ell < \ell'$, since if $\ell' \leq \ell$, we would have

$$(G_\alpha + p^\ell G) \subseteq (G_{\alpha'} + p^{\ell'} G)$$

and it wold be impossible to form two rows of an inp-pattern with the relations $\equiv_{p^\ell,\alpha}$ and $\equiv_{p^{\ell'},\alpha}$.

Now pick $c \in G$ such that

$$c \equiv_{p^\ell,\alpha} t(\overline{a}_{i,0}) \wedge c \equiv_{p^{\ell'},\alpha'} t'(\overline{a}_{i',0}),$$

and pick $d \in G$ such that

$$d \equiv_{p^\ell,\alpha} t(\overline{a}_{i,1}) \wedge d \equiv_{p^{\ell'},\alpha'} t'(\overline{a}_{i',0}).$$

Then

$$c - d \in (G_{\alpha'} + p^{\ell'} G) \setminus (G_\alpha + p^\ell G) \subseteq (G_{\alpha'} + p^\ell G) \setminus (G_\alpha + p^\ell G),$$

and thus

$$G_\alpha \subseteq H_{p^\ell}(c-d) \subsetneq G_{\alpha'}.$$

To finish the proof of Lemma 4.5, we just have to observe that $H_{p^\ell}(c-d)$ is a subgroup named by a sort in $\mathcal{S}_p$. This follows from Lemma 2.2(1) of [3], but we recall the proof here for the sake of completeness. Let $s \in \mathbb{N}$ be maximal such that $c - d \in H_{p^\ell}(c-d) + p^s G$. By definition, $s < \ell$. Write $c - d = b + p^s b'$ for some $b \in H_{p^\ell}(c-d)$ and $b' \in G$, and we claim that $H_{p^\ell}(c-d) = H_p(b')$. On the one hand, b' is not an element of $H_{p^\ell}(c-d) + pG$ since, otherwise, we would have $c - d \in H_{p^\ell}(c-d) + p^{s+1}G$, contradicting the maximality of s. On the other hand, if H is any convex subgroup strictly larger than $H_{p^\ell}(c-d)$, then

$$b + p^s b' = c - d \in H + p^\ell G \subseteq H + p^{s+1} G,$$

so, $p^s b' \in H + p^{s+1} G$, and therefore $b' \in H + pG$, as desired. ☑

Proof of Theorem 1.2: Suppose that the set $\mathbb{P}_{sing}$ of singular primes is finite and that $\mathcal{S}_p$ is finite for each $p \in \mathbb{P}_{sing}$, and that we have an inp-pattern of depth n in a single variable x satisfying all of the assumptions above.

Then at most one row consists of a conjunction of Type (III) formulas, and all other rows consist of single Type (I) formulas of the form $x \equiv_{p^\ell, \alpha} t(\overline{a}_{ij})$ for some $p \in \mathbb{P}_{sing}$. By Lemma 4.5, for each singular prime p, there are at most $|\mathcal{S}_p|$ rows in our inp-pattern. Therefore the total depth of the inp-pattern is at most

$$1 + \sum_{p \in \mathcal{P}_{sing}} |\mathcal{S}_p|$$

and, in particular, the dp-rank of G is finite. □

5. Example

5.1. Optimality of the upper bound in Theorem 1.2

For any prime p, let

$$\mathbb{Z}_{(p)} = \{\frac{a}{b} : a, b \in \mathbb{Z},\, b \neq 0,\, \gcd(b, p) = 1\}.$$

Fix some countably infinite subset $B \subseteq \mathbb{R}$ such that $1 \in B$ and the elements of B are linearly independent over $\mathbb{Q}$, and let G_p be the ordered subgroup of $\mathbb{R}$ consisting of all finite sums $a_1 b_1 + \ldots + a_k b_k$ such that $a_i \in \mathbb{Z}_{(p)}$ and $b_i \in B$. The essential properties of G_p are that it is an Archimedean ordered abelian group, $[G_p : pG_p] = \infty$, and for any prime $q \neq p$, $qG_p = G_p$.

For $j \in \omega \setminus \{0\}$, we let G_p^j stand for the direct sum of j copies of G_p, ordered lexicographically. One can readily check that G_p^j has p as its unique singular prime. One may also check that the dp-rank of G_p^j (in $\mathcal{L}_{oag}$) is $j+1$, although this is not necessary for the ensuing proposition and in fact follows from its proof.

Proposition 5.1. *For any finite sequence of elements $k_0, \dots, k_{m-1} \in \omega \setminus \{0\}$, let $p_0, \dots, p_{m-1}$ denote the first m prime numbers and let*

$$G = \mathbb{Q} \oplus \bigoplus_{i=0}^{m-1} G_{p_i}^{k_i},$$

ordered lexicographically (so that the first coordinate in $\mathbb{Q}$ takes precedence). Then the singular primes for G are $p_0, \dots, p_{m-1}$, for each such p_i the cardinality of $\mathcal{S}_{p_i}$ is k_i, and the dp-rank of G is $1 + \sum_{i=0}^{m-1} k_i$.

Proof. The fact that $p_0, \dots, p_{m-1}$ are the singular primes for G is immediate. On the one hand, for any $i < m$, if we let

$$H_i = \bigoplus_{j>i} G_{p_j}^{k_j},$$

then the sort $\mathcal{S}_{p_i}$ consists of names for the convex subgroups

$$0, G_{p_i} \oplus H_i, G_{p_i}^2 \oplus H_i, \dots, G_{p_i}^{k_i-1} \oplus H_i$$

(as can be checked by the definition of the groups $H_{p_i}(a)$ as a simple exercise); thus $|S_{p_i}| = k_i$, and since $p_0, \dots, p_{m-1}$ are all the singular primes of G, the fact that the dp-rank of G is less than or equal to $1 + \sum_{i=0}^{m-1} k_i$ follows from Theorem 1.2.

On the other hand, to get the opposite rank inequality, we just need to exhibit an inp-pattern of depth $1 + \sum_{i=0}^{m-1} k_i$ in G. To accomplish this, for any $i \in \{0, \dots, m-1\}$ and any $j \in \{0, \dots, k_i - 1\}$, pick elements $\{c_{i,j,k} : k \in \omega\} \subseteq p_i^j G_{p_i}$ which represent distinct cosets of $p_i^{j+1} G_{p_i}$ and let $e_{i,j,k} \in G$ be the element whose coordinate in the jth copy of G_{p_i} (counting from the right) is $c_{i,j,k}$ and all of whose other coordinates are equal to 0.

Finally, we can construct an inp-pattern as follows: for each $i \in \{0, \dots, m-1\}$ and each $j \in \{1, \dots, k_i\}$, construct a row of formulas

$$\varphi_{i,j}(x; e_{i,j,k}) := x \equiv_{p_i^{j+1} \alpha_{i,j}} e_{i,j,k}$$

where $\alpha_{i,j}$ is an element in the sort $\mathcal{S}_{p_i}$ representing the convex subgroup $G_{p_i}^{j-1} \oplus H_i$, unless $j = 1$ in which case we let $\alpha_{i,j}$ be a name for the trivial subgroup $\{0\}$. The final row in the inp-pattern will consist of pairwise disjoint intervals $a_k < x < b_k$ (for $k \in \omega$) constructed by first picking elements $a_0^0 < b_0^0 < a_1^0 < b_1^0 < \dots$ in $\mathbb{Q}$ and then letting a_k, b_k be elements whose $\mathbb{Q}$-coordinate is a_k^0 (or b_k^0, respectively) and all of whose other coordinates are equal to zero.

It is immediate that each row of the pattern described above is 2-inconsistent, and all that is left is to explain why, given any $k \in \omega$ and any choice of function $\eta : A \to \omega$, where A is the set of all pairs (i, j) with $i \in \{0, \dots, m-1\}$ and $j \in \{1, \dots, k_i\}$, there is an element $d \in (a_k, b_k)$ which satisfies

$$d \equiv_{p_i^{j+1}, \alpha_{i,j}} e_{i,j,\eta(i,j)}$$

for all pairs $(i, j) \in A$. For this, we may pick $c_k \in \mathbb{Q}$ such that $a_k < c_k < b_k$, let $d_k \in G$ be such that its $\mathbb{Q}$-coordinate is c_k and all of its other coordinates are 0, and let

$$d = d_k + \sum_{i<m, j<k_i} e_{i,j,\eta(i,j)}.$$

We leave it as an exercise to the reader to verify that this element d satisfies all the required formulas. ☑

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