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A direct proof of a theorem of Jech and Shelah on PCF algebras

Una prueba directa de un teorema de Jech y Shelah sobre álgebras PCF

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Abstract. By using an argument based on the structure of the locally compact scattered spaces, we prove in a direct way the following result shown by Jech and Shelah: there is a family $\{B_\alpha : \alpha < \omega_1\}$ of subsets of ω_1 such that the following conditions are satisfied:

(a) $\max B_\alpha = \alpha$,
(b) if $\alpha \in B_\beta$ then $B_\alpha \subseteq B_\beta$,
(c) if $\delta \leq \alpha$ and δ is a limit ordinal then $B_\alpha \cap \delta$ is not in the ideal generated by the sets $B_\beta, \beta < \alpha$, and by the bounded subsets of δ,
(d) there is a partition $\{A_n : n \in \omega\}$ of ω_1 such that for every α and every n, $B_\alpha \cap A_n$ is finite.

Key words and phrases. PCF theory, locally compact scattered space.

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Resumen. Utilizando un argumento basado en la estructura de los espacios localmente compactos dispersos, demostramos de una manera directa el siguiente resultado de Jech y Shelah: existe una familia $\{B_\alpha : \alpha < \omega_1\}$ de subconjuntos de ω_1 que verifica las siguientes condiciones:

(a) $\max B_\alpha = \alpha$,
(b) si $\alpha \in B_\beta$ entonces $B_\alpha \subseteq B_\beta$,
(c) si $\delta \leq \alpha$ y δ es un ordinal límite, entonces $B_\alpha \cap \delta$ no pertenece al ideal generado por los conjuntos $B_\beta, \beta < \alpha$, y por los subconjuntos acotados de δ,
(d) existe una partición $\{A_n : n \in \omega\}$ de ω_1 tal que para todo α y para todo n, $B_\alpha \cap A_n$ es finito.

Palabras y frases clave. teoría PCF, espacio localmente compacto disperso.

1. Introduction

By Easton's well-known theorem, we have that if V satisfies the Generalized Continuum Hypothesis, then for every monotone function $f : \mathrm{OR} \to \mathrm{OR}$ such that $\alpha < f(\alpha)$ and $\aleph_\alpha < \mathrm{cf}(\aleph_{f(\alpha)})$ for each α there is a cardinal-preserving generic extension of V where $2^{\aleph_\alpha} = \aleph_{f(\alpha)}$ for every ordinal α such that $\aleph_\alpha$ is regular. So, any cardinal arithmetic behaviour satisfying some obvious requirements can be realized as the behaviour of the power function at regular cardinals. However, the freedom enjoyed by the power function on regular cardinals does not extend to singular cardinals. In fact, Shelah proved a series of results getting cardinal bounds on the behaviour of the power function at singular cardinals by studying reduced products of regular cardinals below the concerned singular cardinal. This led to the so called PCF theory, a powerful general tool which has been used to obtain important results in cardinal arithmetic, and which also found interesting applications in algebra and topology (see [1], [3] and [7]).

Recall that an infinite cardinal κ is a *strong limit cardinal* if $2^\lambda < \kappa$ for every cardinal $\lambda < \kappa$. Then, the following remarkable theorem was proved by Shelah in [7].

Theorem 1.1. *If $\aleph_\omega$ is a strong limit cardinal, then $2^{\aleph_\omega} < \aleph_{\omega_4}$.*

Although significant results have been obtained by Gitik, Shelah, Woodin, Magidor and others, it is unknown whether the bound in Theorem 1.1 can be improved to $\aleph_{\omega_3}$, $\aleph_{\omega_2}$ or even to $\aleph_{\omega_1}$.

One of the key objects in PCF theory is the PCF operator, which is defined as follows: if A is a set of regular cardinals, then

$$\mathrm{PCF}(A) = \{\mathrm{cf}(\Pi A/D) : D \text{ is an ultrafilter on } A\}.$$

In order to show Theorem 1.1, Shelah proved that, for $A = \{\aleph_{n+1} : n < \omega\}$, $|\mathrm{PCF}(A)| \leq \omega_3$. A major open problem in the theory of singular cardinals is whether the set $\mathrm{PCF}(A)$ can be uncountable. If we could prove that $\mathrm{PCF}(A)$ is countable, we would improve Shelah's bound on $2^{\aleph_\omega}$ to $\aleph_{\omega_1}$. With respect to this problem, it was shown in [5, Theorem 2.1] that if $\mathrm{PCF}(\{\aleph_{n+1} : n < \omega\})$ is uncountable, then a certain PCF algebra on ω_1 exists. And it is easy to show that this PCF algebra can be obtained directly from the structure on ω_1 which has the properties listed in the abstract. Then, the following theorem is the main result of [5].

Theorem 1.2. *There is a family $\{B_\alpha : \alpha < \omega_1\}$ of subsets of ω_1 such that the following conditions are satisfied:*

(a) *For every $\alpha < \omega_1$, $\max B_\alpha = \alpha$.*

(b) *For all $\alpha, \beta < \omega_1$, if $\alpha \in B_\beta$ then $B_\alpha \subseteq B_\beta$.*

(c) *If $\delta \leq \alpha < \omega_1$ and δ is a limit ordinal then $B_\alpha \cap \delta$ is not in the ideal generated by the sets $B_\beta, \beta < \alpha$, and by the bounded subsets of δ.*

(d) *There is a partition $\{A_n : n \in \omega\}$ of ω_1 such that for every $\alpha < \omega_1$ and every $n < \omega$, $B_\alpha \cap A_n$ is finite.*

In [5], the existence of the family $\{B_\alpha : \alpha < \omega_1\}$ in Theorem 1.2 was shown to be consistent by means of a forcing argument, and then applying a previous general method introduced in [8] it was proved that the existence of that family is a theorem in ZFC.

A direct forcing-free proof of Theorem 1.2 was supplied by Komjáth in [6] by using a pure combinatorial argument. Then, in this paper we will give an alternative direct proof of Theorem 1.2, different from Komjáth's argument and based on the structure of the locally compact scattered spaces.

For every $\alpha < \omega_1$, we put $I_\alpha = \{\omega \cdot \alpha + n : n < \omega\}$. Clearly, $\omega_1 = \bigcup\{I_\alpha : \alpha < \omega_1\}$. We define the functions $\pi : \omega_1 \to \omega_1$ and $\rho : \omega_1 \to \omega$ as follows. Assume that $\delta \in \omega_1$. Then, if $\delta = \omega \cdot \alpha + n$ for $\alpha < \omega_1$ and $n < \omega$, we put $\pi(\delta) = \alpha$ and $\rho(\delta) = n$. We say that a partial order $\preceq$ on ω_1 is *admissible*, if $x \prec y$ implies $\pi(x) < \pi(y)$.

In both direct proofs, the structure described in the statement of Theorem 1.2 is obtained from an admissible partial order on ω_1. In the construction carried out in [6], the required partial order $\leq$ is the transitive closure f^* of a function $f : \omega_1 \to [\omega_1]^{\leq\omega}$ such that for every $x \in \omega_1$, $f(x) \subseteq \bigcup\{I_\alpha : \alpha < \pi(x)\}$. Then, Komjáth's proof of Theorem 1.2 is obtained directly from the following immediate consequence of [6, Theorem 1].

Theorem 1.3. *There is a function f as above satisfying the following conditions:*

(a) *If $y \in f(x)$ then $\rho(y) > \rho(x)$.*

(b) *If $y, y' \in f(x)$ with $y \neq y'$ then $\rho(y) \neq \rho(y')$.*

(c) *If $\beta < \omega_1$, $x \in I_\alpha$ for some $\beta < \alpha < \omega_1$ and Z is a finite subset of $\bigcup\{I_\gamma : \beta < \gamma < \omega_1\}$ such that $x \notin f^*(z)$ for every $z \in Z$, then there are infinitely many $y \in I_\beta$ such that $y \in f(x)$ and $y \notin f^*(z)$ for every $z \in Z$.*

In our direct proof of Theorem 1.2, we will construct an LCS poset on ω_1, which is a notion equivalent to the notion of an SBA ordering given in [4], satisfying some specific properties. In our construction, the required partial order on ω_1 will be defined by transfinite induction without using an auxiliary function f as above. The main difference between both direct constructions is the verification of conditions (c) and (d) in the statement of Theorem 1.2. In [6], the verification of these conditions is carried out by using properties $(a) - (c)$ in the statement of Theorem 1.3. More precisely, in Komjáth's proof, condition (c) of Theorem 1.2 is obtained directly from condition (c) of Theorem 1.3. However, in our construction, condition (c) of Theorem 1.2 is obtained by means of an elementary topological argument applied to the space associated with the LCS poset we construct.

Also, in [6], condition (d) of Theorem 1.2 is verified by using properties (a) and (b) of Theorem 1.3, which are not demanded in our construction. Moreover, in the proof of Theorem 1.3, Komjáth makes use of the fact that his partial order $<$ on ω_1 satisfies that $x < y$ implies $\rho(x) > \rho(y)$. However, this property is not required in the definition of our partial order on ω_1.

2. The direct proof of Theorem 1.2

Recall that a topological space X is *scattered*, if every non-empty subspace of X has an isolated point. By an *LCS space* we mean a locally compact, Hausdorff and scattered space. For an LCS space X and an ordinal α, the *αth-Cantor-Bendixson level* of X is defined by $I_\alpha(X) =$ the set of isolated points of $X \setminus \bigcup\{I_\beta(X) : \beta < \alpha\}$. We define the *height of* X as $\mathrm{ht}(X) =$ the least ordinal α such that $I_\alpha(X) = \varnothing$.

The following notion, which will be used in our proof of Theorem 1.2, permits us to construct in a direct way LCS spaces from partial orders.

Definition 2.1. Assume that $T = \bigcup\{T_\alpha : \alpha < \eta\}$ for some non-zero ordinal η where each T_α is a non-empty set and $T_\alpha \cap T_\beta = \varnothing$ for $\alpha < \beta < \eta$. Assume that for every $x \in T$, b_x is a subset of T such that the following conditions hold:

(1) If $x \in T_\gamma$, then $b_x \cap \bigcup\{T_\xi : \gamma \le \xi < \eta\} = \{x\}$ and $b_x \cap T_\xi$ is infinite for each $\xi < \gamma$.

(2) If $x \in b_y$ then $b_x \subseteq b_y$.

(3) If $x, y \in T$, there are finitely many elements $z_1, \dots, z_n \in T$ such that $b_x \cap b_y = b_{z_1} \cup \cdots \cup b_{z_n}$.

For $x, y \in T$, we put $x \preceq y$ iff $x \in b_y$. Clearly, $\preceq$ is a partial order on T. Then, we will say that $\mathcal{T} = (T, \preceq)$ is an *LCS poset* on T, and we will write $b_x(\mathcal{T}) = b_x$ for every $x \in T$.

Given an LCS poset $\mathcal{T} = (T, \preceq)$ with $T = \bigcup\{T_\alpha : \alpha < \eta\}$, we can topologize T by taking basic open sets to be of the form $b_x \setminus (b_{x_1} \cup \cdots \cup b_{x_n})$ where $n < \omega$ and $x_1, \dots, x_n \prec x$. It can be easily checked that the resulting space $X = X(\mathcal{T})$ is a locally compact, Hausdorff, scattered space such that $\mathrm{ht}(X) = \eta$ and $I_\alpha(X) = T_\alpha$ for every $\alpha < \eta$ (see [2] for a proof). Then, if Y is a subset of T we will denote by $\overline{Y}$ the closure of Y in X. Note that for every $\alpha < \eta$, $\overline{T_\alpha} = \bigcup\{T_\beta : \alpha \le \beta < \eta\}$.

Proof of Theorem 1.2. We construct an LCS poset $\mathcal{T}$ on ω_1 such that the family $\{b_\alpha(\mathcal{T}) : \alpha < \omega_1\}$ satisfies conditions $(a)-(d)$. Recall that $I_\alpha = \{\omega\cdot\alpha + n : n < \omega\}$ for $\alpha < \omega_1$. Then, we will have that $\mathrm{ht}((X(\mathcal{T})) = \omega_1$ and $I_\alpha(X(\mathcal{T})) = I_\alpha$ for every countable α. We write $S_\alpha = \bigcup\{I_\beta : \beta \leq \alpha\}$, and we put $T = \omega_1$. Now, for $n < \omega$ we write $C_n = \{n\} \cup \{\delta + n : \delta$ is a countable limit ordinal$\}$. So, C_n is the n-th column of T.

Then, proceeding by transfinite induction on $\alpha < \omega_1$ we construct for every $x \in I_\alpha$ a subset b_x of S_α satisfying the following conditions:

(1) $b_x \cap I_\alpha = \{x\}$ and $b_x \cap I_\beta$ is infinite for every $\beta < \alpha$.

(2) If $x \in b_y$ then $b_x \subseteq b_y$.

(3) If $x, y \in S_\alpha$, there are finitely many elements $z_1, \ldots, z_n \in S_\alpha$ such that $b_x \cap b_y = b_{z_1} \cup \cdots \cup b_{z_n}$.

(4) If $z \in I_\gamma$ and $\gamma \leq \beta \leq \alpha$, then $\{y \in I_\beta : b_y \cap b_z \neq \varnothing\}$ is finite.

(5) If $m < \omega$ and $\beta \leq \alpha$, then $\{y \in I_\beta : b_y \cap C_m \neq \varnothing\}$ is finite.

(6) For every $x \in S_\alpha$ and every $m < \omega$, $b_x \cap C_m$ is finite.

We put $b_x = \{x\}$ for every $x \in \omega$. Now, assume that $0 < \alpha < \omega_1$ and b_x has been defined for every $x \in \bigcup\{I_\beta : \beta < \alpha\}$. We may assume that α is a limit ordinal. Otherwise, the considerations are similar. We put $Z = \bigcup\{I_\beta : \beta < \alpha\}$. Let $\{\alpha_n : n < \omega\}$ be a strictly increasing sequence of ordinals converging to α. We construct an infinite subset $U = \{u_n : n < \omega\}$ of Z and an infinite subset V of U such that the following conditions hold:

(i) $\bigcup\{b_{u_n} : n < \omega\} = Z$,

(ii) if $u_n \in V$ then $b_{u_n} \cap \bigcup\{b_{u_m} : m < n\} = \varnothing$,

(iii) if $u_n \in V$ then $\alpha_n < \pi(u_n)$,

(iv) if $m < n$ and $u_n \in V$ then $b_{u_n} \cap C_m = \varnothing$.

Let $\{x_m : m < \omega\}$ be an enumeration of Z. Assume that $n \geq 0$ and we have picked the elements $u_0, \ldots, u_{n-1}$. If $n = 2k$ for some $k \geq 0$, we define u_n as the first element u in the enumeration $\{x_m : m < \omega\}$ such that $u \notin \bigcup\{b_{u_m} : m < n\}$. Now, suppose that $n = 2k+1$ for some $k \geq 0$. By conditions (4) and (5), there is an element $u_n \in Z$ with $\pi(u_n) > \max\{\alpha_n, \pi(u_0), \ldots, \pi(u_{n-1})\}$ such that $b_{u_n} \cap \bigcup\{b_{u_m} : m < n\} = \varnothing$ and $b_{u_n} \cap \bigcup\{C_m : m < n\} = \varnothing$. Then, we define $U = \{u_n : n < \omega\}$ and $V = \{v_k : k < \omega\}$ where $v_k = u_{2k+1}$ for $k < \omega$. Clearly, conditions (i)-(iv) hold. Now, let $y_n = \omega\cdot\alpha + n$ for $n < \omega$. Let $\{a_k : k < \omega\}$ be a partition of ω into infinite subsets. Then, we define

$b_{y_k} = \{y_k\} \cup \bigcup\{b_{v_n} : n \in a_k\}$ for $k < \omega$. We can verify that conditions $(1) - (6)$ are satisfied. For this, note that conditions (1) and (2) are obvious, conditions (3) and (4) follow from conditions (i) and (ii), and conditions (5) and (6) follow from condition (iv).

Now, if $x, y \in \omega_1$, we put $x \preceq y$ iff $x \in b_y$. It is obvious that $(T, \preceq)$ is an LCS poset. Let $\mathcal{B} = \{b_x : x \in \omega_1\}$. Clearly, by conditions (1), (2) and (6), $\mathcal{B}$ satisfies conditions $(a), (b)$ and (d) in the statement of the theorem. Now, in order to verify condition (c), assume that $\delta \leq \gamma < \omega_1$ and δ is a limit ordinal. Let $\delta_0 < \delta$ and $\gamma_1, \ldots, \gamma_n < \gamma$. Since δ is a limit, we have that δ is the first element in $I_{\pi(\delta)}$. Hence, as $\delta_0 < \delta \leq \gamma$, we deduce that $\nu = \pi(\delta_0) < \pi(\delta) \leq \pi(\gamma)$. Thus, since $\gamma \in \overline{(I_\nu \setminus \delta_0)}$, we infer that $(b_\gamma \cap I_\nu) \setminus (\delta_0 \cup b_{\gamma_1} \cup \cdots \cup b_{\gamma_n})$ is infinite. So $(b_\gamma \cap \delta) \setminus (\delta_0 \cup b_{\gamma_1} \cup \cdots \cup b_{\gamma_n})$ is infinite too, and hence condition (c) holds. ☑

So, for $x \in \omega_1$, the set b_x in our proof corresponds with the set $f^*(x)$ in Komjáth's approach. A further difference between both direct proofs is that our conditions (3), (4) and (5) are not employed in his construction. In fact, we need condition (3) in order to carry out our topological argument and our conditions (4) and (5) are needed in order to construct the required LCS poset of uncountable height.

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