

A note on the controllability of linear first order systems with holomorphic initial functions in elliptic complex numbers

Una nota sobre la controlabilidad de Sistemas Lineales de Primer Orden con funciones iniciales holomorfas en los Complejos Elípticos

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ABSTRACT. In this paper we characterize the constrained null-controllability of a large family of linear first order systems with holomorphic initial functions in a more general context of holomorphicity given by the elliptic complex numbers.

Key words and phrases. Constrained null-controllability, Initial value problems, Elliptic complex numbers, Strongly continuous semigroups.

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RESUMEN. En este trabajo se caracteriza la controlabilidad restringida en el origen para una familia grande de Sistemas Lineales de Primer Orden con funciones iniciales holomorfas, en un contexto más general de holomorphicidad dado por los números complejos elípticos.

Palabras y frases clave. Controlabilidad restringida en el origen, Problemas a Valor Inicial, Números Complejos Elípticos, Semigrupos Fuertemente Continuos.

1. Introduction

Let us consider the following linear first order systems

$$\partial_t u = a_{11}\partial_x u + a_{12}\partial_y u + a_{21}\partial_x v + a_{22}\partial_y v + c_1 u + c_2 v + c_3 \quad (1)$$

$$\partial_t v = b_{11}\partial_x u + b_{12}\partial_y u + b_{21}\partial_x v + b_{22}\partial_y v + d_1 u + d_2 v + d_3 \quad (2)$$

with $u(t, x, y)$ and $v(t, x, y)$ real-valued functions, t the time and (x, y) running in a bounded domain in the x, y -plane. It is supposed that the coefficients depend at least continuously on t, x and y . By virtue of the Cauchy-Kowaleskaya Theorem the initial value problem

$$u(0, x, y) = \phi(x, y) \quad (3)$$

$$v(0, x, y) = \psi(x, y) \quad (4)$$

is solvable provided the coefficients of (1) and (2) and the initial functions possess power-series representations. In [6] there are given conditions on the coefficients of (1) and (2) under which each initial value problem (3) and (4) is solvable by assuming that the initial functions ϕ and ψ satisfy the Cauchy-Riemann conditions. Those results are generalized in [1], with the assumption that the initial value functions are holomorphic in the sense of an algebra with the structure polynomial $X^2 + \beta X + \alpha$, where α and β are real numbers. In these algebras complex numbers are written as $z = x + iy$ where $i^2 = -\beta i - \alpha$ and the two real functions $u(x, y)$ and $v(x, y)$ satisfy the Cauchy-Riemann equations if $\partial_x u - \alpha \partial_y v = 0$ and $\partial_y u + \partial_x v - \beta \partial_y v = 0$. For details about the above referred algebras see [1] and the references therein.

Let us also recall that a pair of differentiable operators \mathbf{L} and \mathbf{G} is said to be *associated* if $\mathbf{G}(w) = 0$ implies $\mathbf{G}(\mathbf{L}(w)) = 0$, that is if \mathbf{L} sends null-solutions of \mathbf{G} again into null-solutions of \mathbf{G} .

The results in [1] are as follows.

Lemma 1.1. *Suppose α and β are constants satisfying $\alpha\beta^2 - 4\alpha^2 \neq 0$ and A, B, E, F and G are continuously differentiable functions with respect to z and \bar{z} . Then the operator \mathbf{L}*

$$\mathbf{L}w := A\partial_z w + B\overline{\partial_z w} + C\partial_{\bar{z}} w + D\overline{\partial_{\bar{z}} w} + Ew + F\bar{w} + G \quad (5)$$

is associated to the Cauchy-Riemann operator in the complex algebra with structure polynomial $X^2 + \beta X + \alpha$ if and only if B and F are identically equal to zero and A, E and G are holomorphic.

It should be pointed out that the condition $\alpha\beta^2 - 4\alpha^2 \neq 0$ is true for the elliptic case, and also for other cases that are not elliptic.

Then it is shown how systems (1) and (2) can be equivalently written and, by using Lemma 1.1, it is obtained that 10 out of the 14 coefficients depend on the choice of 3 arbitrary holomorphic functions (E, G and A), the other 4 coefficients are free and it is only imposed the condition that they are continuous functions. This is summarized in the following

Lemma 1.2. *Suppose the coefficients $a_{11}, a_{12}, b_{11}, b_{12}$ are arbitrarily chosen. Then the coefficients that characterize the system (1) and (2) are given by: c_i*

and $d_i, i = 1, 2, 3$, and are determined by the choice of 2 arbitrary holomorphic functions (E and G). Then permissible coefficients are given by

$$\begin{aligned} a_{21} &= -\alpha A_2 + a_{12}, & a_{22} &= \alpha A_1 - \alpha a_{11} - \beta a_{12}, \\ b_{21} &= A_1 - \beta A_2 + b_{12}, & b_{22} &= \alpha A_2 - \alpha b_{11} - \beta b_{12} \end{aligned}$$

where $A_1 + iA_2$ is an arbitrary holomorphic function.

Now, for Ω an open connected set in \mathbb{C} and $\{f_n\}_{n=0}^\infty$ a sequence of holomorphic functions that converges to a limit function f (uniformly in every compact subset of Ω), the Weierstrass approximation theorem in the elliptic complex numbers is proven: it is shown that the limit function f is holomorphic in Ω .

From this point on it is possible to solve the initial value problem (1)-(4) in the more general context of holomorphy with respect to the structure polynomial $X^2 + \beta X + \alpha$. It is needed to consider an exhaustion of the bounded domain Ω by a family of subdomains $\Omega_s, 0 < s < s_0$, in such a way that each point x of Ω lies on the boundary $\partial\Omega_{s(x)}$ of a uniquely determined domain $\Omega_{s(x)}$ of the exhaustion. Then $s_0 - s(x)$ is a measure of the distance of a point x of Ω from the boundary $\Omega_{s(x)}$. Let \mathcal{B}_s denote the Banach space of functions which are holomorphic in Ω_s and continuous in $\overline{\Omega_s}$. The $\mathcal{B}_s, 0 < s < s_0$ form a scale of Banach spaces. Our given initial value problem, in its complex version, can now be rewritten as an abstract operator equation in the scale \mathcal{B}_s

$$\partial_t w(t, z) = \mathbf{L}w(t, z), \quad w(0, z) = \rho(z), \tag{6}$$

where $w(t, z) = u(t, z) + iv(t, z), z = x + iy, t$ is the time variable, \mathbf{L} was introduced in Lemma 1.1 and $\rho(z) = \phi(z) + i\psi(z)$ with ϕ, ψ given by (3) and (4) respectively.

Theorem 1.3. *The space of holomorphic functions is associated to the system (1) and (2) if and only if the coefficients of the system are given by Lemma 2. For such systems each initial value problem (3), (4) is solvable, where $\rho = \phi + i\psi$ is an arbitrary holomorphic function in z and the initial functions ϕ and ψ satisfy the generalized Cauchy-Riemann system with respect to the structure polynomial $X^2 + \beta X + \alpha$ with $4\alpha - \beta^2 > 0$. Moreover, the solution, written in the form $w(t, z) = u(t, z) + iv(t, z)$ is holomorphic in z for each t and it exists in the time-interval $0 \leq t \leq h(s_0 - s)$ if h is sufficiently small and (x, y) belongs to Ω_s , where the subdomains Ω_s form an exhaustion of Ω .*

The former theorem was proven by applying a generalized complex abstract Cauchy-Kovaleskaya Theorem (see [7]). This was possible in view of the Weierstrass approximation theorem referred above, and the interior estimate of first order for the complex derivative of a holomorphic function which is obtained by using Cauchy's integral formula. For details see [1].

It is the goal of this note to characterize the null-controllability of the linear first order systems with initial functions described in Theorem 1.3. The control function n is constrained to lie in a non-empty separable, weakly compact subset W of a suitable Banach space N . Bearing in mind this purpose, we have rewritten the given initial value problem as an abstract operator equation.

Now, we will introduce the associated c_0 -semigroup and show that its adjoint is strongly continuous in $(0, \infty)$. All this will be done in the next section.

2. Controllable systems

In order to study the controllability it is necessary to consider non-homogeneous systems (see [4]). We need to suppose that \mathbf{L} is a linear operator in B_s . We have thus the problem

$$(7) \quad \begin{aligned} \partial_t w(t, z) &= \mathbf{L}w(t, z) + K n(t, z) \\ w(0, z) &= \rho(z), \end{aligned}$$

where $B_s, t, z, w(t, z), \mathbf{L}$ and $\rho(z)$ are as in (6), $K : N \rightarrow B_s$ with N a Banach space, is a bounded linear operator and $n : [0, \infty) \times \Omega \rightarrow N$ is a strongly measurable, essentially bounded function.

The existence and uniqueness of a solution for (6) means that there exists a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ in B_s whose infinitesimal generator is \mathbf{L} and thus the mild solution for the non-homogeneous system (7) can be written as

$$w(t, z) = T(t)\rho(z) + \int_0^t T(t-s)K n(s, z) ds.$$

Assuming W a non empty separable, weakly compact subset of N we have

$$W_r = \{n \in L_N^\infty([0, r] \times \Omega) : n \in W \text{ a.e.}\},$$

the set of admissible controls, while

$$A_r(\rho(z)) = \{T(r)\rho(z) + \int_0^r T(r-s)K n(s, z) ds : n \in W_r\}$$

is the set of accesible points. The system (7) will be controllable if $0 \in A_r(\rho(z))$.

We would like to recall that N is an arbitrary Banach space. This gives considerable generality to our results. We also recall that a Banach space M is a Grothendieck space if every weakly*-convergent sequence in M^* is also weakly convergent. Equivalently M is a Grothendieck space if every linear bounded operator from M to any separable Banach space is weakly compact [5]. Among Grothendieck spaces, we can list all reflexive Banach spaces and $L^\infty(\Omega, \Xi, \mu)$ where (Ω, Ξ, μ) is a positive measure space.

A bounded linear operator $T : M \rightarrow N$, where M and N are Banach spaces, factors through a Banach space S if there are bounded linear operators $P : M \rightarrow S$ and $Q : S \rightarrow N$ such that $T = QP$. It is proven in [3]

that if M is a Banach space and $\{T(t)\}_{t \geq 0}$ a C_0 -semigroup defined on M such that for every $a > 0$ there exists a Grothendieck space N_a such that $T(a)$ factors through N_a , then $\{T^*(t)\}_{t \geq 0}$ is strongly continuous on $(0, \infty)$. Factoring through Grothendieck spaces is, in general, not easy to verify, but among semigroups satisfying those assumptions (and hence having adjoints which are strongly continuous on $(0, \infty)$) we mention weakly compact semigroups, i.e., semigroups such that $T(t)$ is weakly compact for each t . This category includes all compact semigroups (see also [3] for details).

Bárceñas and Diestel proved in [2] the following useful controllability criterion: *Let X and U be Banach spaces, let $B : U \rightarrow X$ be a bounded linear operator, and $A : X \rightarrow X$ be the infinitesimal generator of a c_0 -semigroup $\{S(t)\}_{t \geq 0}$ on X whose dual semigroup is strongly continuous on $(0, \infty)$. Suppose Ω is a non-empty separable weakly compact convex subset of U containing 0 . Then for each $T > 0$, $0 \in A_T(x_0)$ if and only if for each $x^* \in X^*$, $\langle x^*, S(T)x_0 \rangle + \int_0^T \max_{v \in \Omega} \langle x^*, S(t)Bv \rangle dt \geq 0$.*

Using that criterion we can characterize the controllability of the system (7). In order to do that we need to verify that the adjoint semigroup $\{T^*(t)\}_{t \geq 0}$ is strongly continuous in $(0, \infty)$. If \mathbf{L} is a bounded linear operator in B_s , then it is well known that $T(t) = e^{\mathbf{L}t} = \sum_{k=0}^{\infty} \frac{(\mathbf{L}t)^k}{k!}$ and it is uniformly continuous.

From this we deduce that $\{T^*(t)\}_{t \geq 0}$ is also uniformly continuous.

If \mathbf{L} is not a bounded operator on B_s , then $\{T(t)\}_{t \geq 0}$ is not uniformly continuous and we cannot assure that $\{T^*(t)\}_{t \geq 0}$ is always strongly continuous on $(0, \infty)$ (although for this case, we can also write $T(t) = e^{\mathbf{L}t}$ for each $t \geq 0$). But assuming that the semigroup $\{T(t)\}_{t \geq 0}$ satisfies that for every $a > 0$ there exists a Grothendieck space N_a such that $T(a)$ factors through N_a (in particular, if $\{T(t)\}_{t \geq 0}$ is a weakly compact semigroup), we can use the above referred result in [3] to obtain that the adjoint semigroup $\{T^*(t)\}_{t \geq 0}$ is strongly continuous in $(0, \infty)$.

To sum up we have proved the following theorem:

Theorem 2.1. *Suppose that the operator \mathbf{L} given by (5) is a linear operator in B_s . If \mathbf{L} is bounded, then for each $r > 0$, $0 \in A_r(\rho(z))$ (i.e., system (7) is controllable) if and only if for each $x^* \in B_s^*$*

$$\langle x^*, T(r)\rho(z) \rangle + \int_0^r \max_{n \in W} \langle x^*, T(t)Kn(t, z) \rangle dt \geq 0.$$

If \mathbf{L} is unbounded, then the same holds if we additionally suppose that $\{T(t)\}_{t \geq 0}$ satisfies the following: for every $a > 0$ there exists a Grothendieck space N_a such that $T(a)$ factors through N_a (in particular, if $\{T(t)\}_{t \geq 0}$ is a weakly compact semigroup).

Remark 2.2. Since $\partial_t w = \mathbf{L}w$ is the rewriting of the system (1) and (2), the permissible coefficients of that system given by Lemma 1.2 depend on the parameters α and β , and N and K are arbitrary, the characterization of the controllability showed in Theorem 2.1 works for a large family of systems (7).

References

- [1] D. Alayón-Solarz and Vanegas C.J, *Operators Associated to the Cauchy-Riemann Operator in Elliptic Complex Numbers*, Advances in Applied Clifford Algebras **22** (2012), no. 2, 257–270.
- [2] D. Bárcenas and J. Diestel, *Constrained controllability in non reflexive banach spaces*, Quaestiones Mathematicae **18** (1995), 185–198.
- [3] D. Bárcenas and L. G. Mármol, *On the adjoint of a strongly continuous semigroup*, Abstract and Applied Analysis **18** (2008), article id 651294.
- [4] R. F. Curtain and H. J. Zwart, *An introduction to infinite-dimensional linear systems theory*, Texts in Applied Mathematics 21, 1995, Springer-Verlag, New York - Berlin.
- [5] J. Diestel, *Grothendieck spaces and vector measures*, contained in Vector and operator valued measures and applications, Proc. Sympos. Alta Utah (1972), 97–108, Academic Press, New York, USA, 1973.
- [6] L. H. Son and W. Tutschke, *First Order differential operators associated to the Cauchy-Riemann equations in the plane*, Complex Variables and Elliptic Equations **48** (2003), no. 9, 797–801.
- [7] F. Trèves, *Basic linear partial differential equations*, New York / San Francisco London, 1975.

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