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Deformations of Noncompact Calabi–Yau threefolds

Deformaciones de Tres-variedades Calabi-Yau no Compactas

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Abstract. We describe deformations of noncompact Calabi–Yau threefolds

$$W_k := \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-k) \oplus \mathcal{O}_{\mathbb{P}^1}(k-2)),$$

for $k = 1, 2, 3$. We compute deformations concretely by calculations of the cohomology group $\mathrm{H}^1(W_k, TW_k)$ via Čech cohomology. We show that for each $k = 1, 2, 3$ the associated structures are qualitatively different, and we also comment on their differences from the analogous structures of simpler noncompact twofolds $\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-k))$.

Key words and phrases. Calabi–Yau, Deformations of noncompact manifolds.

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Resumen. Describimos deformaciones de 3-variedades Calabi-Yau no compactas

$$W_k := \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-k) \oplus \mathcal{O}_{\mathbb{P}^1}(k-2)),$$

para $k = 1, 2, 3$. Concretamente, calculamos las deformaciones a través del primer grupo de cohomología $\mathrm{H}^1(W_k, TW_k)$ vía cohomología de Čech. Mostramos que para cada $k = 1, 2, 3$, las estructuras asociadas son cualitativamente distintas y, además, comentamos sobre sus diferencias con las estructuras análogas de las 2-variedades no compactas $\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-k))$.

Palabras y frases clave. Calabi-Yau, Deformaciones de variedades no compactas.

1. Motivation

Our motivation to study deformations of Calabi–Yau threefolds comes from mathematical physics. In fact, deformations of complex structures of Calabi–Yau threefolds enter as terms of the integrals defining the action of the theories of Kodaira–Spencer gravity [3]. As we shall see, in general our threefolds will have infinite-dimensional deformation spaces, thus allowing for rich applications.

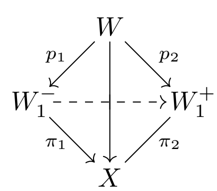
We consider smooth Calabi–Yau threefolds W_k containing a line $\ell \cong \mathbb{P}^1$. For the applications we have in mind for future work it will be useful to observe the effect of contracting the line to a singularity. The existence of a contraction of ℓ imposes heavy restrictions on the normal bundle [6], namely $N_{\ell/W}$ must be isomorphic to one of

$$\text{(a)}\ \ \mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1)\ ,\quad \text{(b)}\ \ \mathcal{O}_{\mathbb{P}^1}(-2)\oplus\mathcal{O}_{\mathbb{P}^1}(0)\ ,\ \ \text{or}\ \ \text{(c)}\ \ \mathcal{O}_{\mathbb{P}^1}(-3)\oplus\mathcal{O}_{\mathbb{P}^1}(+1)\ .$$

Conversely, Jiménez [6] states that if $\mathbb{P}^1 \cong \ell \subset W$ is any subspace of a smooth threefold W such that $N_{\ell/W}$ is isomorphic to one of the above, then:

- in (a) ℓ always contracts,
- in (b) either ℓ contracts or it moves, and
- in case (c) there exists an example in which ℓ does not contract nor does any multiple of ℓ (i.e. any scheme supported on ℓ) move.

W_1 is the space appearing in the basic flop. Let X be the cone over the ordinary double point defined by the equation $xy - zw = 0$ on $\mathbb{C}^4$. The basic flop is described by the diagram:



Here $W := W_{x,y,z,w}$ is the blow-up of X at the vertex $x = y = z = w = 0$, $W_1^- := Z_{x,z}$ is the small blow-up of X along $x = z = 0$ and $W_1^+ := Z_{y,w}$ is the small blow-up of X along $y = w = 0$. The basic flop is the rational map from W^- to W^+. It is famous in algebraic geometry for being the first case of a rational map that is not a blow-up.

Thus, we will focus on the Calabi–Yau cases

$$W_k := \mathrm{Tot}\big(\mathcal{O}_{\mathbb{P}^1}(-k) \oplus \mathcal{O}_{\mathbb{P}^1}(k-2)\big) \text{ for } k = 1, 2, 3.$$

We will also consider surfaces of the form

$$Z_k := \mathrm{Tot}\big(\mathcal{O}_{\mathbb{P}^1}(-k)\big)$$

for comparison in Sections 4 and 5.

2. Statements of results

We describe deformations of complex surfaces and threefolds which are the total spaces of vector bundles on the complex projective line $\mathbb{P}^1$. We focus on the case of Calabi–Yau threefolds. Even though there is no well established theory of deformations for noncompact manifolds, we obtain deformations working by analogy with Kodaira theory for the compact case, see [7]. Namely, we calculate cohomology with coefficients in the tangent bundle, and then proceed to show that the directions of infinitesimal deformations parametrized by first cohomology are integrable, see Section 3.

In the case of surfaces Z_k, with $k > 0$, we prove that the deformations of the surfaces Z_k, described in [2], can be obtained from the deformations of the Hirzebruch surfaces $\mathbb{F}_k$, Lemma 5.5.

Our results on deformations of the threefolds W_k are as follows. We show that W_1 is formally rigid, Theorem 6.1, whereas W_2 has an infinite-dimensional deformation space, Theorem 6.3. Furthermore, we exhibit a deformation $\mathcal{W}_2$ of W_2 which turns out to be a non-affine manifold, a very different case from that of surfaces Z_k, $k > 0$, where all the deformations are affine varieties. Finally, we give an infinite-dimensional family of deformations of W_3 which is not universal, but is semiuniversal, Corollary 6.13. The case W_3 is quite different from W_1, W_2, or the surfaces. The tools used here to describe deformation spaces were not sufficient for W_3, therefore we must look for more effective techniques. The cases $k \geq 3$ present similar features; we will continue their study in future work.

3. Deformations of noncompact manifolds

In this section we describe our methods to find infinitesimal deformations of noncompact manifolds. When looking for deformations of noncompact manifolds one needs to keep in mind the caveat that cohomology calculations are generally not enough to decide questions of existence of infinitesimal deformations, as the following example illustrates.

Example 3.1. Edoardo Ballico gave us the following illustration that cohomological rigidity does not imply absence of deformations.

We consider deformations of $X = \mathbb{C}$. Clearly $\mathrm{H}^1(X, TX) = 0$. However, there do exist nontrivial deformations of X as the following family shows.

Consider $\pi\colon \mathbb{P}^1 \times D \to D$ with D any smooth manifold (even $\mathbb{P}^1$ or a disc) and a specific $o \in D$. Take $s_\infty\colon D \to \mathbb{P}^1 \times D$ the section of π defined by

$$s_\infty(x) = (\infty, x),$$

then take another section s of π with

$$s(o) = (\infty, o), \qquad s(x) = (a_x, x)$$

with $a_x \in \mathbb{C} = \mathbb{P}^1 \setminus \{\infty\}$. Take as the total space for our family Y $\mathbb{P}^1 \times D$ minus the images of the two sections. Then we obtain a deformation of $\mathbb{C}$ in which at all points of $D \setminus \{o\}$ you have $\mathbb{C} \setminus \{0\}$, thus not a trivial deformation in any reasonable sense. Hence, vanishing of cohomology does not imply nonexistence of deformations. Nevertheless, cohomology calculations are useful to find deformations.

In this work, by deformation we mean the following:

Definition 3.2. A *deformation* of a complex manifold X is a holomorphic fiber bundle $\tilde{X} \xrightarrow{\pi} D$, where D is a complex disc centered at 0 (possibly a vector space, possibly infinite dimensional), satisfying:

- $\pi^{-1}(0) = X$,
- $\tilde{X}$ is trivial in the C^∞ (but not necessarily in the holomorphic) category.

***Remark* 3.3.** Our choice for the dimension of D is $n = h^1(X, TX)$ whenever possible. The case $n = 0$ corresponds to the following definition.

Definition 3.4. We call a manifold X *formally rigid* when $\mathrm{H}^1(X, TX) = 0$.

We show in 6.1 that W_1 is formally rigid.

Definition 3.5. We call a manifold X *rigid* if any deformation $\tilde{X} \xrightarrow{\pi} D$ is biholomorphic to the trivial bundle $X \times D \to D$.

In general, formally rigid does not imply rigid. With Definition 3.2 we do not claim to solve the problem that a manifold X does not deform under the condition $\mathrm{H}^1(X, TX) = 0$, however we eliminate some unwanted cases such as the one in Example 3.1.

Observe that the deformations considered in [2] satisfy Definition 3.2, hence maintain the C^∞ type of the manifold. Moreover, for these surfaces, all deformations are affine.

We show that $\mathrm{H}^1(W_2, TW_2) \neq 0$ and then prove that directions of deformations parametrized by such cohomology are integrable by explicitly constructing families. The details for other threefolds will remain for future work.

Note that since X is covered by 2 affine (Stein) open sets, second cohomologies with coherent coefficients vanish, hence there are no obstructions to deformations.

4. Comparison with the deformation theory of surfaces

Deformations of the surfaces Z_k were described in [2]. It turned out rather interestingly that the results we obtained for threefolds are not at all analogous to the ones for surfaces.

Regarding applications to mathematical physics, the deformations of surfaces turned out rather disappointing, because instantons on Z_k disappear under a small deformation of the base [2, Thm. 7.3]. This resulted from the fact that deformations of Z_k are affine varieties. The case of threefolds is a lot more promising, since for $k > 1$, W_k has deformations which are not affine.

Nevertheless, deformations of the surfaces Z_k turned out to have an interesting application to a question motivated by the Homological Mirror Symmetry conjecture: [1, Sec. 2] showed that the adjoint orbit of $\mathfrak{sl}(2,\mathbb{C})$ has the complex structure of the nontrivial deformation of Z_2 and used this structure to construct a Landau–Ginzburg model that does not have projective mirrors. Further applications to mirror symmetry give us another motivation to study deformation theory for Calabi–Yau threefolds.

5. Z_k, their bundles and deformations

In this section we obtain properties about the surfaces Z_k that will be used in the development of the theory of threefolds.

5.1. A holomorphic bundle on $Z_{(-1)}$ that is not algebraic

By definition $Z_{(-1)} = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(+1))$, and in canonical coordinates $Z_{(-1)} = U \cup V$, where $U = \{(z,u)\}$ and $V = \{(\xi,v)\}$, $U \cap V \cong \mathbb{C}^* \times \mathbb{C}$, with change of coordinates given by:

$$\boxed{(\xi, v) \mapsto (z^{-1}, z^{-1}u)}\ .$$

Notation 1. We denote by $\mathcal{O}_{Z_{(-1)}}(-j) = p^*(\mathcal{O}_{\mathbb{P}^1}(-j))$ the pullback bundle of $\mathcal{O}_{\mathbb{P}^1}(-2)$, where $p\colon Z_{(-1)} \to \mathbb{P}^1$ is the natural projection.

Lemma 5.1. $\mathrm{H}^1(Z_{(-1)}, \mathcal{O}_{Z_{(-1)}}(-2))$ *is infinite-dimensional over* $\mathbb{C}$. *It consists of holomorphic functions of the form*

$$\sum_{l \leq -1} \sum_{i \geq 0} a_{li} z^l u^i$$

such that $l + i + 2 > 0$.

Proof. A 1-cocycle σ can be written in the form

$$\sigma = \sum_{i=0}^{+\infty} \sum_{l=-\infty}^{+\infty} \sigma_{i,l} z^l u^i\ .$$

Since monomials containing nonnegative powers of z are holomorphic in U, these are coboundaries, thus

$$\sigma \sim \sum_{i=0}^{+\infty} \sum_{l=-\infty}^{-1} \sigma_{i,l} z^l u^i ,$$

where $\sim$ denotes cohomological equivalence. Changing coordinates, we obtain

$$T\sigma = z^2 \sum_{i=0}^{+\infty} \sum_{l=-\infty}^{-1} \sigma_{i,l} z^l u^i = \sum_{i=0}^{+\infty} \sum_{l=-\infty}^{-1} \sigma_{i,l} z^{l+2} u^i = \sum_{i=0}^{+\infty} \sum_{l=-\infty}^{-1} \sigma_{i,l} \xi^{-l-2-i} v^i ,$$

where terms satisfying $-l-2-i \geq 0$ are holomorphic on V.

Thus, the nontrivial terms on $\mathrm{H}^1(Z_{(-1)}, \mathcal{O}_{Z_{(-1)}}(-2))$ are

$$\begin{array}{ccccc} z^{-1} & z^{-1}u & z^{-1}u^2 & z^{-1}u^3 & \cdots \\ & z^{-2}u & z^{-2}u^2 & z^{-2}u^3 & \cdots \\ & & z^{-3}u^2 & z^{-3}u^3 & \cdots \\ & & & z^{-4}u^3 & \cdots \\ & & & & \ddots \end{array}$$

☑

Proposition 5.2. *The bundle E over $Z_{(-1)}$ defined in canonical coordinates by the matrix*

$$\begin{bmatrix} z^1 & z^{-1}e^u \\ 0 & z^{-1} \end{bmatrix} \tag{1}$$

is holomorphic but not algebraic.

Proof. This bundle E can be represented by the element

$$z^{-1}e^u \in \mathrm{Ext}^1(\mathcal{O}_{Z_{(-1)}}(1), \mathcal{O}_{Z_{(-1)}}(-1)) \simeq \mathrm{H}^1(Z_{(-1)}, \mathcal{O}_{Z_{(-1)}}(-2)).$$

We have

$$\begin{bmatrix} z^1 & z^{-1}e^u \\ 0 & z^{-1} \end{bmatrix} = \begin{bmatrix} z^1 & z\sigma \\ 0 & z^{-1} \end{bmatrix} \tag{2}$$

with $z^{-2}e^u = \sigma \in \mathrm{H}^1(Z_{(-1)}, \mathcal{O}_{Z_{(-1)}}(-2))$, see [5, p. 234]. Observe that

$$\begin{aligned} z^{-2}e^u &= z^{-2}\left(1 + u + \frac{u^2}{2} + \cdots + \frac{u^n}{n!} + \cdots\right) \\ &= z^{-2} + \underbrace{z^{-2}\left(u + \frac{u^2}{2} + \frac{u^3}{6} + \cdots + \frac{u^n}{n!} + \cdots\right)}_{(\gamma)} , \end{aligned}$$

where the monomials in γ represent pairwise distinct nontrivial classes in $\mathrm{H}^1(Z_{(-1)}, \mathcal{O}_{Z_{(-1)}}(-2))$ as shown in Lemma 5.1. Consequently, the class $z\sigma \in \mathrm{Ext}^1(\mathcal{O}_{Z_{(-1)}}(1), \mathcal{O}_{Z_{(-1)}}(-1))$ corresponding to the bundle E cannot be represented by a polynomial, hence E is holomorphic but not algebraic. ☑

Corollary 5.3. *The threefold W_3 has holomorphic bundles that are not algebraic.*

Proof. Consider the map $p\colon W_3 \to Z_{(-1)}$ given by projection on the first and third coordinates, that is, in canonical coordinates as in (7) we see $Z_{(-1)}$ as cut out inside W_3 by the equation $u_1 = 0$. Then the pullback bundle p^*E is holomorphic but not algebraic on W_3. In fact, the same proof works as in Proposition 5.2. ☑

5.1.1. A similar bundle on Z_1

It is instructive to verify the result of defining a bundle by the same matrix, but over the surface Z_1 instead. Recall that $Z_1 = U \cup V$, with change of coordinates given by

$$\boxed{(\xi, v) \mapsto (z^{-1}, zu)}\,.$$

Consider the bundle E on Z_1, given by transition matrix

$$\begin{bmatrix} z^1 & z^{-1}e^u \\ 0 & z^{-1} \end{bmatrix}. \tag{3}$$

Note that this is the same matrix used in (1). Thus E corresponds to the element $z^{-1}e^u \in \mathrm{Ext}^1(\mathcal{O}_{Z_1}(1), \mathcal{O}_{Z_1}(-1)) \simeq \mathrm{H}^1(Z_1, \mathcal{O}_{Z_1}(-2))$. Consequently, we may rewrite the transition function

$$\begin{bmatrix} z^1 & z^{-1}e^u \\ 0 & z^{-1} \end{bmatrix} = \begin{bmatrix} z^1 & z\sigma \\ 0 & z^{-1} \end{bmatrix} \tag{4}$$

where $z^{-2}u = \sigma \in H^1(Z_1, \mathcal{O}_{Z_1}(-2))$. But $\sigma = \xi^3 v$ is holomorphic on the V chart, and hence a coboundary. Thus $\sigma = 0 \in \mathrm{H}^1(Z_1, \mathcal{O}_{Z_1}(-2))$, and accordingly $z^{-1}e^u = 0 \in \mathrm{Ext}^1(\mathcal{O}_{Z_1}(1), \mathcal{O}_{Z_1}(-1))$. Therefore the extension splits and

$$E = \mathcal{O}_{Z_1}(-1) \oplus \mathcal{O}_{Z_1}(1).$$

In [4], Gasparim proved that every holomorphic bundle on Z_k is algebraic with $k \geq 1$.

5.2. Deformations of Z_k

Recall that a family is semiuniversal (in the sense of [8, Def. 1.35]) if the Kodaira-Spencer map is bijective. In [2, Thm. 5.3], Barmeier and Gasparim constructed a $(k-1)$-dimensional semiuniversal deformation space $\mathcal{Z}$ for Z_k given by

$$(\xi, v, t_1, \ldots, t_{k-1}) = (z^{-1}, z^k u + t_{k-1} z^{k-1} + \cdots + t_1 z, t_1, \ldots, t_{k-1}). \tag{5}$$

We now prove that this family fits our definition of deformation.

Lemma 5.4. *The deformation given by eq. 5 is a C^∞-trivial fiber bundle.*

Proof. Note that for any C^∞ function $f\colon U \to \mathbb{C}$, the manifold given by gluing the charts $V = \mathbb{C}^2_{\xi,v}$ and $U = \mathbb{C}^2_{z,u}$ by

$$(\xi, v, t_1, \ldots, t_{k-1}) = (z^{-1}, z^k u + f(z,u))$$

whenever $z \neq 0$ and $\xi \neq 0$ is diffeomorphic to Z_k.

We have that z^{-1} and $z^k u$ is C^∞. Then u is C^∞, as well as $\mathrm{Re}(u)$ and $\mathrm{Im}(u)$, respectively the real and imaginary parts of u. Hence

$$\frac{zu + z\bar{u}}{2\mathrm{Re}(u)} \text{ and } \frac{zu - z\bar{u}}{2i\mathrm{Im}(u)}$$

are C^∞ and coincide with z whenever $\mathrm{Re}(u)$ and $\mathrm{Im}(u)$ are not equal to 0, respectively. We define then

$$f(z,u) = \begin{cases} \dfrac{zu + z\bar{u}}{2\mathrm{Re}(u)}, \mathrm{Re}(u) \neq 0 \\ \dfrac{zu - z\bar{u}}{2i\mathrm{Im}(u)}, \mathrm{Im}(u) \neq 0 \\ z, u = 0 \end{cases},$$

which is C^∞ on the intersection. Furthermore, f coincides with z.

We conclude that $g(z,u) = t_{k-1} z^{k-1} + \cdots + t_1 z, t_1, \ldots, t_{k-1}$ is C^∞. ☑

Lemma 5.5. *Deformations of Z_k can be obtained from deformations of $\mathbb{F}_k$. Thus, the family $\mathcal{Z}$ is is not universal.*

Proof. We compare deformations of the surfaces Z_k with those of the Hirzebruch surfaces. Choose coordinates $(t_1, \ldots, t_{k-1}, [l_0, l_1], [x_0, \ldots, x_{k+1}])$ for the product $\mathbb{C}^{k-1}_t \times \mathbb{P}^1_l \times \mathbb{P}^{k+1}_x$. [8, Chap. II] shows that the Hirzebruch surface $\mathbb{F}_k$ has a $(k-1)$-dimensional semiuniversal deformation space given by the smooth subvariety $M \subset \mathbb{C}^{k-1}_t \times \mathbb{P}^1_l \times \mathbb{P}^{k+1}_x$ cut out by the equations

$$l_0(x_1, x_2, \ldots, x_k) = l_1(x_2 - t_1 x_0, \ldots, x_k - t_{k-1} x_0, x_{k+1}) \;. \tag{6}$$

Let $\mathcal{Z}$ and M denote the deformations given by 5 and 6, respectively. Now consider the following map:

$$\begin{array}{rcl} f\colon \mathcal{Z} & \to & M \\ (z,u,t_1,\ldots,t_{k-1}) & \mapsto & (t_1,\ldots,t_{k-1},[1,z],[-1,z_1,\ldots,z_k,u]) \\ (\xi,v,t_1,\ldots,t_{k-1}) & \mapsto & (t_1,\ldots,t_{k-1},[\xi,1],[-1,v,\xi_2,\ldots,\xi_{k+1}]) \end{array}$$

where we used the following notation:

$$\begin{array}{ll} z_1 = z^k u + t_{k-1}z^{k-1} + \cdots + t_1 z & \xi_2 = \xi v - t_1 \\ z_2 = z^{k-1}u + t_{k-1}z^{k-2} + \cdots + t_2 z & \xi_3 = \xi^2 v - t_1\xi - t_2 \\ \vdots & \vdots \\ z_{k-1} = z^2 u + t_{k-1}z & \xi_k = \xi^{k-1}v - t_1\xi^{k-2} - \cdots - t_{k-1} \\ z_k = zu & \xi_{k+1} = \xi^k v - t_1\xi^{k-1} - \cdots - t_{k-1}\xi \end{array}$$

It turns out that this map is injective and satisfies $f(\mathcal{Z}_t) \subset M_t$ for all $t \in \mathbb{C}^{k-1}$. Notice that, for each $t \in \mathbb{C}^{k-1}$, we can decompose M_t as

$$M_t = A_t \cup B_t \ ,$$

where $A_t = \{p \in M_t, x_0 = 0\}$ and $B_t = \{p \in M_t, x_0 \neq 0\}$. It then follows that

- $B_t = f(\mathcal{Z}_t)$, and
- A_t is the boundary of B_t,

implying as a corollary that: $M_t = M_{t'}$ if and only if $\mathcal{Z}_t = \mathcal{Z}_{t'}$.

So we conclude that each Z_k has as many deformations as $\mathbb{F}_k$, specifically, $\lfloor k/2 \rfloor$. In particular, $\mathcal{Z}_k$ is not universal. ☑

6. Deformations of Calabi–Yau threefolds

6.1. Rigidity of W_1

Theorem 6.1. ([9]) W_1 *is formally rigid.*

***Proof*.** Formal infinitesimal deformations of complex structures are parameterized by first cohomology with coefficients in the tangent bundle. Direct calculation of Čech cohomology shows that $\mathrm{H}^1(W_1, TW_1) = 0$. Hence W_1 is formally rigid, Definition 3.4. ☑

6.2. Deformations of W_2

Since we have $\mathrm{H}^2(W_2, TW_2) = 0$, we can make an analogy with unobstructed deformations in the compact case, where the theorem of existence [7, Thm. 5.6] guarantees integrability of the cocycles in $\mathrm{H}^1(W_2, TW_2)$. This theorem does not apply in the noncompact case. For the case that we consider, we will prove existence by explicitly constructing the corresponding manifold as Lemma 6.5 shows.

It is possible to obtain some deformations using compactifications, in which case we can use the well developed theory of deformations from [7]. However, given the results of Theorem 6.3, infinitely many directions of deformations of W_2 would be lost if we worked with the compactification. Hence, we favor an approach using Definition 3.2.

For instance, suppose we consider the compactification of W_2 given by:

$$\overline{W}_2 = \mathrm{Proj}\big(\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}\big).$$

Lemma 6.2. *$\overline{W}_2$ has only two directions of deformation.*

Proof. The first cohomology group of $\overline{W}_2$ is isomorphic to $\mathbb{C}^2$ as a vector space, that is, $\mathrm{H}^1(\overline{W}_2, T\overline{W}_2) = \mathbb{C}^2$. ☑

In fact, many non affine deformations would remain unfound with this method.

Theorem 6.3. ([9]) *W_2 has an infinite-dimensional family of deformations.*

Proof. The proof will follow from Lemmas 6.4 and 6.5 below. First we show that the first cohomology with tangent coefficients is infinite-dimensional. Then we show that its cocycles are integrable, and thus they parameterize deformations of W_2. ☑

Lemma 6.4. *$\mathrm{H}^1(W_2, TW_2)$ is infinite dimensional over $\mathbb{C}$. If consists of holomorphic sections of the form*

$$\sum_{j\geq 0} \begin{bmatrix} 0 \\ z^{-1}u_2^j \\ 0 \end{bmatrix}$$

(written in canonical coordinates).

Proof. W_2 can be covered by

$$U = \{(z, u_1, u_2)\} \quad \text{and} \quad V = \{(\xi, v_1, v_2)\}\ ,$$

with $U \cap V = \mathbb{C} - \{0\} \times \mathbb{C} \times \mathbb{C}$ and transition function given by

$$\boxed{(\xi, v_1, v_2) = (z^{-1}, z^2 u_1, u_2)}\,.$$

We have then that the transition function for TW_2 is

$$A = \begin{bmatrix} -z^{-2} & 0 & 0 \\ 2zu_1 & z^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let σ be a 1-cocycle, i.e. a holomorphic function on $U \cap V$:

$$\sigma = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{l=-\infty}^{\infty} \begin{bmatrix} a_{lij} \\ b_{lij} \\ c_{lij} \end{bmatrix} z^l u_1^i u_2^j\ .$$

But

$$\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty} \begin{bmatrix} a_{lij} \\ b_{lij} \\ c_{lij} \end{bmatrix} z^l u_1^i u_2^j$$

is a coboundary, so

$$\sigma \sim \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{l=-\infty}^{-1} \begin{bmatrix} a_{lij} \\ b_{lij} \\ c_{lij} \end{bmatrix} z^l u_1^i u_2^j = \sigma',$$

where $\sim$ denotes cohomological equivalence. So

$$\begin{aligned} A\sigma' &= \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{l=-\infty}^{-1} \begin{bmatrix} -a_{lij} z^{-2} \\ 2a_{lij} z u_1 + b_{lij} z^2 \\ c_{lij} \end{bmatrix} z^l u_1^i u_2^j \\ &= \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{l=-\infty}^{-1} \begin{bmatrix} -a_{lij} z^{-4} \\ 2a_{lij} z^{-3}(z^2 u_1) + b_{lij} \\ c_{lij} z^{-2} \end{bmatrix} z^{2+l-2i} (z^2 u_1)^i u_2^j \\ &= \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{l=-\infty}^{-1} \begin{bmatrix} -a_{lij} \xi^4 \\ 2a_{lij} \xi^3 v_1 + b_{lij} \\ c_{lij} \xi^2 \end{bmatrix} \xi^{2i-l-2} v_1^i v_2^j. \end{aligned}$$

Except for the case where $l = -1$ and $i = 0$, we have that $2i - l - 2 \geq 0$, thus the corresponding monomials are holomorphic in V and hence coboundaries.

It follows that

$$A\sigma' \sim \sum_{j=0}^{\infty} \begin{bmatrix} -a_j\xi^4 \\ 2a_j\xi^3 v_1 + b_j \\ c_j\xi^2 \end{bmatrix} \xi^{-1} v_2^j$$
$$\sim \sum_{j=0}^{\infty} \begin{bmatrix} 0 \\ b_j \\ 0 \end{bmatrix} \xi^{-1} v_2^j,$$

where we omit the indices -1 for l and 0 for i for simplicity. We conclude then that $\mathrm{H}^1(W_2, TW_2)$ is infinite-dimensional, generated by the sections

$$\sigma_j = \begin{bmatrix} 0 \\ z^{-1}u_2^j \\ 0 \end{bmatrix}$$

for $j \geq 0$. ☑

Lemma 6.5. *All cocycles in* $\mathrm{H}^1(W_2, TW_2)$ *are integrable.*

Proof. We can write the transition function of W_2 as:

$$\begin{bmatrix} \xi \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} z^{-1} \\ z^2 u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} z^{-2} & 0 & 0 \\ 0 & z^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z \\ u_1 \\ u_2 \end{bmatrix}.$$

As we computed in Lemma 6.4, $\mathrm{H}^1(W_2, TW_2)$ is generated by the sections

$$\begin{bmatrix} 0 \\ z^{-1}u_2^j \\ 0 \end{bmatrix}$$

for $j \geq 0$. Then we can express the deformation family for W_2 as

$$\begin{bmatrix} \xi \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} z^{-2} & 0 & 0 \\ 0 & z^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} z \\ u_1 \\ u_2 \end{bmatrix} + \sum_{j\geq 0} t_j \begin{bmatrix} 0 \\ z^{-1}u_2^j \\ 0 \end{bmatrix} \right)$$
$$= \begin{bmatrix} z^{-1} \\ z^2 u_1 + \sum_{j\geq 0} t_j z u_2^j \\ u_2 \end{bmatrix},$$

i.e. we have an infinite-dimensional deformation family given by

$$U = \mathbb{C}^3_{z,u_1,u_2} \times \mathbb{C}[t_j] \quad \text{and} \quad V = \mathbb{C}^3_{\xi,v_1,v_2} \times \mathbb{C}[t_j]$$

with

$$(\xi, v_1, v_2, t_0, t_1, \dots) = \left(z^{-1}, z^2 u_1 + \sum_{j \geq 0} t_j z u_2^j, u_2, t_0, t_1, \dots \right)$$

on the intersection $U \cap V = (\mathbb{C} - \{0\}) \times \mathbb{C}^2 \times \mathbb{C}[t_j]$. ☑

The proof that this family is C^∞ trivial is similar to the proof of Lemma 5.4.

6.2.1. A non-affine deformation

The proof of Lemma 6.5 gives us that deformations of W_2 are threefolds given by change of coordinates of the form

$$(\xi, v_1, v_2) = \left(z^{-1}, z^2 u_1 + \sum_{j \geq 0} t_j z u_2^j, u_2 \right).$$

We consider now the example $\mathcal{W}_2$ that occurs when $t_1 = 1$ and all t_j vanish for $j \neq 1$, that is, the one with change of coordinates

$$(\xi, v_1, v_2) = \left(z^{-1}, z^2 u_1 + z u_2, u_2 \right).$$

Lemma 6.6. *Let $\mathcal{O}_{\mathcal{W}_2}(-j) = p^*(\mathcal{O}_{\mathbb{P}}(-j))$ denote the pullback bundle of $\mathcal{O}_{\mathbb{P}}(-j)$, where $p\colon \mathcal{W}_2 \to \mathbb{P}$ is the natural projection. Then $\mathrm{H}^1(\mathcal{W}_2, \mathcal{O}_{\mathcal{W}_2}(-4)) \neq 0$.*

Proof. Consider the 1-cocycle σ written in the U coordinate chart as $\sigma = z^{-1}$. Suppose σ is a coboundary, then we must have

$$\sigma = \alpha + T^{-1}\beta$$

where $\alpha \in \Gamma(U)$ and $\beta = \Gamma(V)$. Consequently

$$z^{-1} = \alpha(z, u_1, u_2) + z^{-4}\beta(z^{-1}, z^2 u_1 + z u_2, u_2)\ .$$

But α has only positive powers of z, and the highest power of z appearing on $z^{-4}\beta$ is -4, hence the right-hand side has no terms in z^{-1} and the equation is impossible, a contradiction. ☑

Corollary 6.7. *$\mathcal{W}_2$ is not affine.*

***Remark* 6.8.** Note that this result contrasts with the situation for surfaces, since [2, Thm. 6.15] proves that all nontrivial deformations of Z_k are affine.

***Remark* 6.9.** The referee pointed out that all deformations of W_2 such that $t_0 = 0$ are affine since they contain a $\mathbb{P}^1$.

6.3. Deformations of W_3

We start by computing the group $\mathrm{H}^1(W_3, TW_3)$ which parameterizes formal infinitesimal deformations of W_3. Recall that W_3 can be covered by $U = \{(z, u_1, u_2)\}$ and $V = \{(\xi, v_1, v_2)\}$, with $U \cap V = \mathbb{C} - \{0\} \times \mathbb{C}^2$ and transition function given by:

$$\boxed{(\xi, v_1, v_2) = (z^{-1}, z^3 u_1, z^{-1} u_2)} \tag{7}$$

Theorem 6.10. *There is a semiuniversal deformation space $\mathcal{W}$ for W_3 parametrised by cocycles of the form*

$$\begin{bmatrix} a_{lij} \\ b_{lij} \\ c_{lij} \end{bmatrix} z^l u_1^i u_2^j \qquad 3i - 3 - l - j < 0.$$

Proof. In canonical coordinates, the transition matrix for the tangent bundle TW_3 is given by

$$T = \begin{bmatrix} -z^{-2} & 0 & 0 \\ 3z^2 u_1 & z^3 & 0 \\ -z^{-2} u_2 & 0 & z^{-1} \end{bmatrix} \simeq \begin{bmatrix} z^{-1} & 0 & -z^{-2} u_2 \\ 0 & z^3 & 3z^2 u_1 \\ 0 & 0 & -z^{-2} \end{bmatrix}, \tag{8}$$

where $\simeq$ denotes isomorphism, and the latter expression is handier for calculations. A 1-cocycle can be expressed in U coordinates in the form

$$\begin{aligned} \sigma &= \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{l=-\infty}^{\infty} \begin{bmatrix} a_{lij} \\ b_{lij} \\ c_{lij} \end{bmatrix} z^l u_1^i u_2^j \\ &\sim \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{l=-\infty}^{-1} \begin{bmatrix} a_{lij} \\ b_{lij} \\ c_{lij} \end{bmatrix} z^l u_1^i u_2^j, \end{aligned}$$

where $\sim$ denotes cohomological equivalence. Changing coordinates we obtain

$$T\sigma = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{l=-\infty}^{-1} \begin{bmatrix} a_{lij} z^{-1} - c_{lij} z^{-2} u_2 \\ 3a_{lij} z^2 u_1 + b_{lij} z^3 \\ -c_{lij} z^{-2} \end{bmatrix} z^l u_1^i u_2^j$$

where all terms inside the matrix are holomorphic on V except for

$$\begin{bmatrix} 0 \\ b_{lij} z^3 \\ 0 \end{bmatrix}.$$

These impose the condition for a cocycle to be nontrivial. Since we have

$$z^3 z^l u_1^i u_2^j = z^{l+3-3i+j}(z^3u_1)^i(z^{-1}u_2)^j = \xi^{3i-3-l-j}u_1^i u_2^j \ ,$$

a nontrivial cocycle satisfies $3i-3-l-j<0$. ☑

We now give a partial description of deformations of W_3.

Lemma 6.11. *The sections*

$$\sigma_1 = \begin{bmatrix} 0 \\ z^{-1} \\ 0 \end{bmatrix} \text{ and } \sigma_2 = \begin{bmatrix} 0 \\ z^{-2} \\ 0 \end{bmatrix}$$

are nonzero cocycles on $\mathrm{H}^1(W_3, TW_3)$.

Proof. Let

$$\sigma_l = \begin{bmatrix} o \\ z^{-l} \\ 0 \end{bmatrix}$$

for $l=1,2$. Then σ_l is not a coboundary on the chart U. We change coordinates by multiplying by the transition T given in 8,

$$T\sigma_l = \begin{bmatrix} 0 \\ z^{l+3} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \xi^{-l-3} \\ 0 \end{bmatrix},$$

which is not holomorphic on the chart V and therefore not a coboundary. ☑

Lemma 6.12. *The following 2-parameter family of deformations of* W_3 *is contained in* $\mathcal{W}$*:*

$$(\xi, v_1, v_2) = (z^{-1}, z^3u_1 + t_2z^2 + t_1z, z^{-1}u_2)$$

Proof. The transition for W_3 is given by,

$$(\xi, v_1, v_2) = (z^{-1}, z^3u_1, z^{-1}u_2).$$

In matrix form:

$$\begin{bmatrix} \xi \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} z^{-2} & 0 & 0 \\ 0 & z^3 & 0 \\ 0 & 0 & z^{-1} \end{bmatrix} \begin{bmatrix} z \\ u_1 \\ u_2 \end{bmatrix}.$$

So we can construct a deformation family for W_3 using the cocycles from Lemma 6.11:

$$\begin{bmatrix} \xi \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} z^{-2} & 0 & 0 \\ 0 & z^3 & 0 \\ 0 & 0 & z^{-1} \end{bmatrix} \left(\begin{bmatrix} z \\ u_1 \\ u_2 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ z^{-1} \\ 0 \end{bmatrix} + t_1 \begin{bmatrix} 0 \\ z^{-2} \\ 0 \end{bmatrix} \right)$$
$$= \begin{bmatrix} z^{-1} \\ z^3 u_1 + t_2 z^2 + t_1 z \\ z^{-1} v_2 \end{bmatrix}$$

Now it suffices to observe that, by Lemma 6.11, σ_1 and σ_2 are nontrivial directions in $\mathcal{W}$. ☑

Corollary 6.13. *The family presented in Theorem 6.12 is formally semiuniversal but not universal.*

Proof. As a consequence of Lemma 6.12 and Corollary 5.5, we have that the deformations in the directions of the cocycles of Lemma 6.11 are isomorphic. Indeed, these deformations are induced by Z_3 which, as $\mathbb{F}_3$, only has one nontrivial direction of deformation. ☑

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References

[1] E. Ballico, S. Barmeier, E. Gasparim, L. Grama, and L.A.B. San Martin, *A Lie theoretical construction of a Landau-Ginzburg model without projective mirrors*, arXiv:1610.06965.

[2] S. Barmeier and E. Gasparim, *Classical deformations of local surfaces and their moduli spaces of instantons*, arXiv:1604.01133. (2017).

[3] M. Bershadsky, *Kodaira-Spencer theory of gravity*, Quantum Field Theory and String Theory **328** (1995), 23–38.

[4] E. Gasparim, *Holomorphic bundles on $\mathcal{O}(-k)$ are algebraic*, Communications in Algebra **25(9)** (1997), 3001–3009.

[5] R. Hartshorne, *Algebraic Geometry*, graduate texts in mathematics, no.52. ed., Springer-Verlag, New York, 1977.

[6] J. Jiménez, *Contraction of nonsingular curves*, Duke Math. **65** (1992), 313–332.

[7] K. Kodaira, *Complex manifolds and deformations of complex structures*, Berlin: Springer, 1986.

[8] M. Manetti, *Lectures on deformations of complex manifolds (deformations from differential graded viewpoint)*, vol. 7, Rend. Mat. Appl., 2004.

[9] F. Rubilar, *Deformaciones de estructuras complejas de 3-variedades Calabi-Yau*, Tesis de magíster, Universidad Católica del Norte, Antofagasta, Chile, marzo 2017.

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