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Orthogonal Decomposition in Omega-Weighted Classes of Functions Subharmonic in the Half-Plane

Descomposición ortogonal de funciones subharmónicas en el semiplano por medio de clases omega-pesadas

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Abstract. The paper gives a harmonic, ω-weighted, half-plane analog of W. Wirtinger's projection theorem and its $(1-r)^{\alpha}$-weighted extension by M. Djrbashian and also an orthogonal decomposition for some classes of functions subharmonic in the half-plane.

Key words and phrases. Subharmonic functions, orthogonal decomposition, potentials.

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Resumen. El artículo da un análogo armónico ω-pesado en el semiplano del teorema de proyección de W. Wirtinger y su extensión $(1-r)^{\alpha}$-pesada establecida por M. Djrbashian. También es hallada una descomposición ortogonal para algunas clases de funciones subarmónicas en el semiplano.

Palabras y frases clave. Funciones subarmónicas, descomposición ortogonal, potenciales.

1. Introduction

The present paper gives a harmonic, ω-weighted, half-plane analog of the Wirtinger projection theorem [8] (see also [7], p. 150) and its $(1-r)^{\alpha}$-weighted extension by M. Djrbashian (see Theorem VII in [1]), which are for holomorphic in $|z|<1$ functions with square integrable modules. These results are a continuation of the results of [5] in the half-plane. Then, an orthogonal decomposition is found for some classes of functions subharmonic in the upper half-plane, which is similar to the result of [4] in the unit disc.

After a useful remark, we shall introduce the spaces of functions which we consider.

***Remark* 1.1.** It is well-known (see, eg. [6], Ch. VI) that the Hardy space h^p $(1 \leq p < +\infty)$ of real, harmonic in the upper half-plane $G^+ := \{z : \mathrm{Im}\ z > 0\}$ functions, defined by the condition

$$\|u\|_{h^p} := \sup_{y>0} \left\{ \int_{-\infty}^{+\infty} |u(x+iy)|^p dx \right\}^{1/p} < +\infty,$$

is a Banach space, becoming a Hilbert space for $p = 2$. Since $|u|^p$ is subharmonic in G^+ for any function u harmonic in G^+, the results of Ch. 7 in [2] on the equivalent definition of the holomorphic Hardy spaces H^p in G^+ have their obvious analogs for h^p. In particular, the space h^p $(1 \leq p < +\infty)$ coincides with the set of all functions harmonic in G^+ and such that

$$\|u\|_{h^p}^p = \lim_{R\to+\infty} \liminf_{y\to+0} \int_{-R}^{R} |u(x+iy)|^p dx < +\infty$$

and, for sufficiently small values of $\rho > 0$,

$$\liminf_{R\to+\infty} \frac{1}{R} \int_{\beta}^{\pi-\beta} \left|u\left(Re^{i\vartheta}\right)\right|^p \left(\sin \frac{\pi(\vartheta - \beta)}{\pi - 2\beta}\right)^{\frac{\pi+2\beta}{\pi-2\beta}} d\vartheta = 0, \tag{1}$$

where $\beta = \arcsin(\rho/R)$. Note that due to Hölder's inequality, if (1) is true for some $p > 1$, then it is true also for $p = 1$.

Definition 1.2. $\widetilde{\Omega}_\alpha$ $(-1 < \alpha < +\infty)$ is the set of functions ω which are continuous, strictly increasing in $[0, +\infty)$, continuously differentiable in $(0, +\infty)$ and such that $\omega(0) = 0$ and $\omega'(x) \asymp x^\alpha$, $\Delta < x < +\infty$, for some $\Delta > 0$.

Definition 1.3. For any $\omega \in \widetilde{\Omega}_\alpha$ $(-1 < \alpha < +\infty)$, h^p_ω $(0 < p < +\infty)$ is the set of the real, harmonic in the upper half-plane G^+ functions for which (1) is true along with

$$\|u\|_{p,\omega} := \left\{ \iint_{G^+} |u(z)|^p d\mu_\omega(z) \right\}^{1/p} < +\infty, \tag{2}$$

where $d\mu_\omega(x+iy) = dx d\omega(2y)$.

2. Some Properties of the Spaces h^p_ω

First, we prove that the above introduced classes h^p_ω are Banach spaces.

Proposition 2.1. h^p_ω $(1 \leq p < +\infty,\ \omega \in \widetilde{\Omega}_\alpha,\ \alpha > -1)$ *is a Banach space with the norm* (2), *which for* $p = 2$ *becomes a Hilbert space with the inner product*

$$(u, v)_\omega := \frac{1}{2\pi} \iint_{G^+} u(z)v(z) d\mu_\omega(z), \quad u, v \in h^2_\omega.$$

Proof. Let L^p_ω $(1 \le p < +\infty)$ be the Banach space of real functions, defined solely by (2). Then, it suffices to prove that h^p_ω is a closed subspace of L^p_ω for any $1 \le p < +\infty$, i.e. if a sequence $\{u_n\}_1^\infty \subset h^p_\omega$ converges to some $u \in L^p_\omega$ in the norm of L^p_ω, then $u \in h^p_\omega$. To this end, observe that

$$\int_0^{1/2} d\omega(2y) \int_{-\infty}^{+\infty} |u_n(x+iy) - u(x+iy)|^p dx \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, by Fatou's lemma we have $\int_0^1 g(t) d\omega(t) = 0$ for

$$g(2y) := \liminf_{n\to\infty} \int_{-\infty}^{+\infty} \big|u_n(x+iy) - u(x+iy)\big|^p dx. \tag{3}$$

As $\omega \in \widetilde{\Omega}_\alpha$, there exists a sequence $\eta_k \downarrow 0$ such that $\omega(\eta_{k+1}) < \omega(\eta_k)$. Introducing the measure $\nu(E) = \bigvee_E \omega$ (i.e. the variation of ω on the set E), we conclude that $\nu([\eta_{k+1}, \eta_k]) > 0$ for any $k \ge 1$ and obviously $g(t) = 0$ in $[\eta_{k+1}, \eta_k]$ almost everywhere with respect to the measure ν. On the other hand, $u(x+it) \in L^p(-\infty, +\infty)$ for almost every $t > 0$ with respect to the measure ν. Thus, there is a sequence $y_k \downarrow 0$ such that simultaneously $g(2y_k) = 0$ and $u(x+iy_k) \in L^p(-\infty, +\infty)$. Now, we choose a subsequence of $\{u_n\}_1^\infty$, for which the limit (3) is attained for $y = y_1$. From this subsequence, we choose another one, for which (3) is attained for $y = y_2$, etc. Then, by a diagonal operation we choose a subsequence for which we keep the same notation $\{u_n\}_1^\infty$, and by which

$$g(2y_k) = \lim_{n\to\infty} \int_{-\infty}^{+\infty} \big|u_n(x+iy_k) - u(x+iy_k)\big|^p dx = 0 \tag{4}$$

for all $k \ge 1$. Then, in virtue of Remark 1.1, for any $n \ge 1$ and $\rho > 0$ the function $u_n(z+i\rho)$ belongs to h^p. Note that in particular this is so for $\rho = y_k$ $(k = 1, 2, \ldots)$. By (4), for any fixed $k \ge 1$ the sequence $\{u_n(z+iy_k)\}_{n=1}^\infty$ is fundamental in h^p, and consequently $u_n(z+iy_k) \to U(z+iy_k) \in h^p$ as $n \to \infty$ in the norm of h^p over G^+. Hence, u_n uniformly tends to U inside G^+, and $U \in h^p$ in any half-plane G^+_ρ. Thus, by the results of Ch. 7 in [2] we conclude that (1) is true for U and, in addition, for any number $A > 0$

$$\iint_{\substack{|x|<A \\ \frac{1}{A}<y<A}} \big|U(z) - u(z)\big|^p d\mu_\omega(z) \le 2^{p-1} \left\{ \iint_{\substack{|x|<A \\ \frac{1}{A}<y<A}} \big|U(z) - u_n(z)\big|^p d\mu_\omega(z) \right.$$
$$\left. + \iint_{\substack{|x|<A \\ \frac{1}{A}<y<A}} \big|u(z) - u_n(z)\big|^p d\mu_\omega(z) \right\} \to 0 \quad \text{as} \quad n \to \infty.$$

The passage $A \to +\infty$ gives $\|U - u\|_{L^p_{\omega,\gamma}} = 0$. ☑

Now, let us prove a theorem on an explicit form of the orthogonal projection of the space L^2_ω to its harmonic subspace h^2_ω. Assuming that $\omega \in \widetilde{\Omega}_\alpha$, $\alpha > -1$, we shall deal with the Cauchy-type kernel

$$C_\omega(z) := \int_0^{+\infty} e^{itz} \frac{dt}{I_\omega(t)}, \quad I_\omega(t) := \int_0^{+\infty} e^{-tx} d\omega(x),$$

which is a holomorphic function in G^+ [3]. Note that by Lemma 3.1 of [3] *for any* $\omega \in \widetilde{\Omega}_\alpha$ *with* $\alpha > -1$ *and any numbers* $\rho > 0$ *and a noninteger* $\beta \in ([\alpha]-1, \alpha)$ *there exists a constant* $M_{\rho,\beta} > 0$ *such that*

$$|C_\omega(z)| \le \frac{M_{\rho,\beta}}{|z|^{2+\beta}}, \quad z \in G^+_\rho := \{z : \text{Im } z > \rho\}. \tag{5}$$

Under the same assumption, we use the Green type potentials constructed by means of the elementary Blaschke type factor

$$b_\omega(z,\zeta) := \exp\left\{ \int_0^{2\text{Im } \zeta} C_\omega(z-\zeta+it)\omega(t)dt \right\}, \quad \text{Im } z > \text{Im } \zeta > 0$$

(see formula (23) in [5]), which is a holomorphic function in G^+, where it has a unique, simple zero at $z = \zeta$.

Theorem 2.2. *If* $\omega \in \widetilde{\Omega}_\alpha$ $(-1 < \alpha < +\infty)$, *then the orthogonal projection of* L^2_ω *to* h^2_ω *can be written in the form*

$$P_\omega u(z) = \frac{1}{\pi} \iint_{G^+} u(w) \text{Re}\{C_\omega(z-\overline{w})\} d\mu_\omega(w), \quad z \in G^+. \tag{6}$$

***Proof*.** Let $u \in L^2_\omega$. Then, applying the estimate (5), where $\beta = \alpha - \varepsilon$ with a small $\varepsilon > 0$, and Hölder's inequality, one can be convinced that the integral of (6) is absolutely and uniformly convergent inside G^+, and hence it represents a harmonic function there. Besides, using the estimate (5) and Hölder's inequality one can prove that for any fixed $\rho > 0$ and $\varepsilon > 0$ small enough there exists a constant $M'_{\rho,\varepsilon} > 0$ depending only on ρ and ε, such that $\left|P_\omega u(Re^{i\vartheta})\right|^2 \le M'_{\rho,\varepsilon} R^{-(3+2\alpha-2\varepsilon)}$ $\left(\arcsin \frac{\rho}{R} < \vartheta < \pi - \arcsin \frac{\rho}{R}\right)$ for $R > 0$. Hence, $P_\omega u$ satisfies (1). Thus, it remains to show that P_ω is a bounded operator which maps L^2_ω to h^2_ω and is identical on h^2_ω.

If $u \in L^2_\omega$, then for a fixed $z = x + iy \in G^+$ and $\zeta = \xi + i\eta$

$$\begin{aligned} P_\omega u(z) &= \text{Re}\left\{ \frac{1}{\pi} \int_0^{+\infty} \left(\lim_{R\to+\infty} \int_{-R}^{R} u(\zeta)d\xi \int_0^{+\infty} e^{it(z-\overline{\zeta})} \frac{dt}{I_\omega(t)} \right) d\omega(2\eta) \right\} \\ &= \text{Re}\left\{ \frac{1}{\pi} \int_0^{+\infty} \left(\lim_{R\to+\infty} \int_0^{+\infty} e^{itz} \frac{e^{-t\eta}}{I_\omega(t)} dt \int_{-R}^{R} e^{-t\xi} u(\zeta) d\xi \right) d\omega(2\eta) \right\} \\ &= \text{Re}\left\{ \frac{1}{\sqrt{\pi}} \int_0^{+\infty} d\omega(2\eta) \int_0^{+\infty} e^{itz} \frac{e^{-t\eta}}{I_\omega(t)} \widehat{u_\eta}(t) dt \right\}, \end{aligned} \tag{7}$$

where the limit $\widehat{u_\eta}(t) = \text{l.i.m.}_{R\to+\infty}\int_{-R}^{R} e^{-it\xi}u(\xi+i\eta)d\xi$ in the $L^2(-\infty,+\infty)$-norm is the Fourier transform of $u(\xi+i\eta)\in L^2(-\infty,+\infty)$ for almost every $\eta>0$. Note that the equalities in (7) are true, since by Plancherel's theorem

$$\frac{1}{\sqrt{\pi}}\int_0^{+\infty}\frac{e^{-t\eta}}{I_\omega(t)}\left|\frac{1}{\sqrt{2\pi}}\int_{-R}^{R}e^{-it\xi}u(\zeta)d\xi-\widehat{u_\eta}(t)\right|dt$$
$$\le \left[C_{\widetilde{\omega}}(2i\eta)\right]^{1/2}\left\|\frac{1}{\sqrt{\pi}}\int_{-R}^{R}e^{-it\xi}u(\zeta)d\xi-\widehat{u_\eta}(t)\right\|_{L^2(-\infty,+\infty)}\to 0$$

as $R\to+\infty$, where the function $\widetilde{\omega}$ is the Volterra square of ω (see Lemma 4 in [5]). From (7) we conclude that

$$P_\omega u(z) = \text{Re}\left\{\frac{1}{\sqrt{\pi}}\int_0^{+\infty}e^{itz}\frac{\Phi(t)}{\sqrt{I_\omega(t)}}dt\right\},\quad z\in G^+, \tag{8}$$

where

$$\Phi(t) := \frac{1}{\sqrt{I_\omega(t)}}\int_0^{+\infty}e^{-t\eta}\widehat{u_\eta}(t)d\omega(2\eta). \tag{9}$$

The change of the integration order transforming (7) to (8) is valid, since by (5) for a fixed $y>0$ and a small $\varepsilon>0$ there is a constant $M>0$ such that

$$\frac{1}{\sqrt{\pi}}\int_0^{+\infty}d\omega(2\eta)\int_0^{+\infty}\frac{e^{-t(y+\eta)}}{I_\omega(t)}|\widehat{u_\eta}(t)|dt$$
$$\le\sqrt{2}\int_0^{+\infty}\left[C_{\widetilde{\omega}}(2i(y+\eta))\right]^{1/2}\|\widehat{u_\eta}\|_{L^2(0,+\infty)}d\omega(2\eta)$$
$$\le M\sqrt{2}\|u\|_{L^2_\omega}\left(\int_0^{+\infty}\frac{d\omega(2\eta)}{(y+\eta)^{3+2\alpha-\varepsilon}}\right)^{1/2}<+\infty,$$

where $\widetilde{\omega}$ is the Volterra square of ω (see Lemma 4 in [5]). By an application of Hölder's inequality and Plancherel's theorem, from (9) we get $\|\Phi\|_{L^2(0,+\infty)}\le\sqrt{2}\|u\|_{L^2_\omega}$, while by the Paley-Wiener theorem (see eg. [6], pp. 130-131) from (9) we obtain

$$\|P_\omega u\|^2_{L^2_\omega}\le\frac{1}{\pi}\int_0^{+\infty}d\omega(2y)\int_0^{+\infty}e^{-2yt}\frac{|\Phi(t)|^2}{I_\omega(t)}dt = 2\|\Phi\|^2_{L^2(0,+\infty)}.$$

Thus, P_ω is a bounded operator which maps L^2_ω to h^2_ω.

Now, let $u\in h^2_\omega$. Then obviously $u(z+i\eta)\in h^2$ for any $\eta>0$. Hence, for any fixed $\eta>0$ the function $u(z+i\eta)$ is the real part of some function $f(z+i\eta)$ from the holomorphic Hardy space H^2 in G^+. Consequently, by the Paley-Wiener Theorem

$$f(z+i\eta)=\frac{1}{\sqrt{2\pi}}\int_0^{+\infty}e^{itz}\widehat{f}_\eta(t)dt,\quad z\in G^+,$$

where the limit by norm

$$\widehat{f}_\eta(t) = \underset{R\to+\infty}{\text{l.i.m.}} \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} e^{-it\xi} f(\xi + i\eta) d\xi \tag{10}$$

is the Fourier transform of f on the level $i\eta$, and

$$\|f(\xi + i\eta)\|^2_{L^2(-\infty,+\infty)} = \|f(z + i\eta)\|^2_{H^2} = \|\widehat{f}_\eta\|^2_{L^2(0,+\infty)}.$$

Note that one can prove the independence of the function $e^{t\eta}\widehat{f}_\eta(t)$ of $\eta > 0$. Further, for any $\eta > 0$ and $\zeta = \xi + i\eta$

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{+\infty} u(\xi + i\eta) C_\omega(z - \overline{\zeta}) d\xi &= \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{i(z+i\eta)t} \widehat{u}_\eta(t) \frac{dt}{tI_\omega(t)} \\ &= \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} e^{i(z+i\eta)t} \left[\widehat{f}_\eta(t) + \widehat{\overline{f}}_\eta(t)\right] \frac{dt}{tI_\omega(t)}.\end{aligned}$$

From (10) and the Paley-Wiener theorem, it follows that for $t > 0$,

$$0 = \overline{\widehat{f}_\eta(-t)} = \underset{R\to+\infty}{\text{l.i.m.}} \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} e^{-it\xi} \overline{f(\xi + i\eta)} d\xi = \widehat{\overline{f}}_\eta(t).$$

Consequently, for any $z \in G^+_\eta$

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} u(\xi + i\eta) C_\omega(z - \overline{\zeta}) d\xi = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{i(z+i\eta)t} \widehat{f}_\eta(t) \frac{dt}{tI_\omega(t)}$$

and hence,

$$\begin{aligned}P_\omega u(z) &= \mathrm{Re}\left\{\frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{izt} \frac{dt}{tI_\omega(t)} \int_0^{+\infty} e^{-2t\eta} \{e^{t\eta}\widehat{f}_\eta(t)\} d\omega(2\eta)\right\} \\ &= \mathrm{Re}\left\{\frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{i(z-i\eta)t} \widehat{f}_\eta(t) dt\right\} = \mathrm{Re}\{f(z)\} = u(z),\end{aligned}$$

i.e. the operator P_ω is an identity on h^2_ω. ☑

3. Orthogonal Decomposition

In virtue of Remark 2 and Theorem 2 in [5], if $\omega \in \widetilde{\Omega}_\alpha$ $(\alpha > -1)$ and ν is the associated Riesz measure of a subharmonic in G^+ function $U \in L^1_\omega$ satisfying (1) with $p = 2$, then

$$\iint_{G^+} \left(\int_0^{2\mathrm{Im}\,\zeta} \omega(t) dt\right) d\nu(\zeta) < +\infty \quad \text{and} \quad \iint_{G^+_\rho} \mathrm{Im}\,\zeta \, d\nu(\zeta) < +\infty$$

for any $\rho > 0$, conditions which provide the convergence of the potential

$$P_\omega(z) = \iint_{G^+} \log|b_\omega(z,\zeta)|d\nu(\zeta)$$

in G^+, and U is representable in the form

$$\begin{aligned} U(z) &= \iint_{G^+} \log|b_\omega(z,\zeta)|d\nu(\zeta) + \frac{1}{\pi}\iint_{G^+} U(w)\big\{\mathrm{Re}\ C_\omega(z-\overline{w})\big\}d\mu_\omega(w) \\ &:= G_\omega(z) + u_\omega(z), \quad z \in G^+. \end{aligned} \tag{11}$$

The next theorem gives an orthogonal decomposition for some ω-weighted classes of functions subharmonic in G^+.

Theorem 3.1. *If $\omega \in \widetilde{\Omega}_\alpha$ with $-1 < \alpha < +\infty$, then:*

(1) Both summands G_ω and u_ω in the right-hand side of the representation (11) of any function $U \in L^2_\omega \cap L^1_\omega$ satisfying (1) with $p = 2$ are of L^2_ω.

(2) The operator P_ω is an identity on h^2_ω and it maps all Green type potentials $G_\omega \in L^1_\omega$ satisfying (1) with $p = 2$ to identical zero.

(3) Any harmonic function $u \in h^2_\omega$ is orthogonal in L^2_ω to any Green type potential $G_\omega \in L^1_\omega \cap L^2_\omega$ satisfying (1) with $p = 2$.

Proof. Let $U \in L^1_\omega \cap L^2_\omega$ be a function which is subharmonic in G^+ and satisfies (1) with $p = 2$. Then, U is representable in the form (11), where $u \in h^2_\omega$ by Theorem 2.2. Hence, also $G_\omega \in L^2_\omega$ and satisfies (1) with $p = 2$. Further, if $G_\omega \in L^1_\omega$ and satisfies (1) with $p = 2$, then applying the operator P_ω to both sides of the equality (11) written for G_ω we get $P_\omega G_\omega(z) \equiv 0$, $z \in G^+$. Since P_ω is the orthogonal projection of L^2_ω to its harmonic subspace h^2_ω, we conclude that

$$\big(P_\omega U, G_\omega\big)_\omega = \big(P_\omega u, G_\omega\big)_\omega = \big(P^*_\omega u, G_\omega\big)_\omega = \big(u, P_\omega G_\omega\big)_\omega = 0.$$

At last, if u is a function of h^2_ω and a Green type potential $G_\omega \in L^1 \cap L^2$ and satisfies (1) with $p = 2$, then by Theorem 2.2

$$\big(u, G_\omega\big)_\omega = \big(P_\omega u, G_\omega\big)_\omega = \big(P^*_\omega u, G_\omega\big)_\omega = \big(u, P_\omega G_\omega\big)_\omega = 0.$$

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