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A sandwich theorem and stability result of Hyers-Ulam type for harmonically convex functions

Un teorema del sándwich y un resultado de estabilidad de tipo Hyers-Ulam para funciones armónicamente convexas

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ABSTRACT. We prove that the real functions f and g, defined on a real interval $[a, b]$, satisfy

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tg(y) + (1-t)g(x),$$

for all $x, y \in [a, b]$ and $t \in [0, 1]$ iff there exists a harmonically convex function $h : [a, b] \to \mathbb{R}$ such that $f(x) \leq h(x) \leq g(x)$ for all $x \in [a, b]$. We also obtain an approximate convexity result, namely we prove a stability result of Hyers-Ulam type for harmonically convex functions.

***Key words*:** Harmonically convex functions, Sandwich theorem, Hyers-Ulam.

RESUMEN. Demostramos que las funciones reales f y g, definidas en un intervalo $[a, b]$, satisfacen

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tg(y) + (1-t)g(x)$$

para todo $x, y \in [a, b]$ y $t \in [0, 1]$ si y sólo si existe una función armónicamente convexa $h : [a, b] \to \mathbb{R}$ tal que $f(x) \leq h(x) \leq g(x)$ para cada $x \in [a, b]$. También obtenemos un resultado de aproximación convexa, es decir, se demuestra un resultado de estabilidad del tipo de Hyers-Ulam para funciones armónicamente convexas.

***Palabras clave*:** Funciones armónicamente convexas, teorema del sándwich, Hyers-Ulam.

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1. Introduction

Due to its important role in mathematical economics, engineering, management science, and optimization theory, convexity of functions and sets has been studied intensively; see [3, 5, 7, 9, 10, 11, 16, 18, 19] and the references therein. Consequently, the classical concept of convex function has been extended and generalized in different directions.

Most important generalizations can be found in works that change the standard requirements for a function to be convex, thereby introducing new notions such as being quasi-convex (see [8]), pseudo-convex (see [1]), strongly convex [24], approximately convex [4], midconvex (see [25]), h-convex [27] , etc.

In this article, we are dealing with a recent notion of generalized convexity, introduced by I. ISCAM in [16], where he gave the following definition of harmonically convex functions.

Definition 1 ([16]). Let I be an interval in $\mathbb{R}-\{0\}$. A function $f: I \to \mathbb{R}$ is said to be *harmonically convex* on I if the inequality

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y)+(1-t)f(x), \tag{1}$$

holds, for all $x, y \in I$ and $t \in [0,1]$.

Hence, harmonically convex functions relate the harmonic mean of two points to the arithmetic mean of the function values at the two points.

Proposition 1 ([16]). *Let $I \subseteq \mathbb{R}\backslash\{0\}$ be a real interval and $f: I \to \mathbb{R}$ a function. Then:*

- *If $I \subseteq (0,+\infty)$ and f is convex and nondecreasing, then f is harmonically convex.*
- *If $I \subseteq (0,+\infty)$ and f is harmonically convex and nonincreasing, then f is convex.*
- *If $I \subseteq (-\infty,0)$ and f is harmonically convex and nondecreasing, then f is convex.*
- *If $I \subseteq (-\infty,0)$ and f is convex and nonincreasing, then f is harmonically convex.*

For some recent results, investigations, and extensions of harmonically convex functions interested readers are referred to [6, 14, 15, 16, 22, 23, 28].

In [6] we can find the following simple but important fact.

Theorem 1 ([6]). *If $[a,b] \subset I \subseteq (0,+\infty)$ and we consider the function $g: \left[\frac{1}{b}, \frac{1}{a}\right] \to \mathbb{R}$ defined by $g(t) = f\left(\frac{1}{t}\right)$, then f is harmonically convex on $[a,b]$ if and only if g is convex in the usual sense on $\left[\frac{1}{b}, \frac{1}{a}\right]$.*

It is easy to verify that this result is satisfied if we use the interval $(0, +\infty)$ rather than the interval $[a, b]$.

This theorem is very important, because it tells us that the graph of the function $f\left(\frac{1}{x}\right)$, in the interval $\left[\frac{1}{b}, \frac{1}{a}\right]$, is located below the line segment $y = ab\frac{f(a)-f(b)}{b-a}\left(x - \frac{1}{a}\right)$.

Moreover, if a function f is harmonically convex then it satisfies the following inequalities, for $x_1 \leq x_3 \leq x_2$:

$$x_2x_3\frac{f(x_2)-f(x_3)}{x_2-x_3} \quad \leq \quad x_1x_2\frac{f(x_2)-f(x_1)}{x_2-x_1} \leq x_1x_3\frac{f(x_3)-f(x_1)}{x_3-x_1},$$

which could be considered as a harmonically perturbed convexity.

The following theorem on separation of functions can be found in the seminal papers of BARON et.al. [2], where the authors proved the following sandwich theorem.

Theorem 2 ([2]). *Two real functions f and g defined on a real interval I satisfy*

$$f(tx + (1-t)y) \leq tg(x) + (1-t)g(y) \tag{2}$$

for all $x, y \in I$ and $t \in [0, 1]$, if and only if there exists a convex function $h : I \to \mathbb{R}$ such that $f \leq h \leq g$.

2. Main results

In this paper we have two main results. The first one is a sandwich theorem for harmonically convex functions, a result that is related to the theorem on separation by convex functions presented in [2]. As a second contribution, we obtain an approximate convexity result, namely, we prove a stability result of Hyers-Ulam type for harmonically convex functions.

Theorem 3. *Let f, g be real functions defined on the interval $(0, +\infty)$. The following conditions are equivalent:*

(i) there exists a harmonically convex function $h : (0, +\infty) \to \mathbb{R}$ such that $f(x) \leq h(x) \leq g(x)$, for all $x \in (0, +\infty)$.

(ii) the following inequalities hold:

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq tg(y) + (1-t)g(x), \tag{3}$$

for all $x, y \in (0, +\infty)$ and $t \in [0, 1]$.

Proof. [(i) $\Rightarrow$(ii)] We assume that there is a harmonically convex function $h : (0, +\infty) \to \mathbb{R}$ such that $f(x) \leq h(x) \leq g(x)$, for all $x \in (0, +\infty)$.

We consider the functions $F, G, H : (0,+\infty) \to \mathbb{R}$ defined by

$$F(x) := f\left(\frac{1}{x}\right), \qquad G(x) := g\left(\frac{1}{x}\right) \qquad \text{and} \qquad H(x) := h\left(\frac{1}{x}\right).$$

Note that, by Theorem 1, H is a function that is convex on $(0,+\infty)$ and satisfies the inequality $F\left(\frac{1}{x}\right) \leq H\left(\frac{1}{x}\right) \leq G\left(\frac{1}{x}\right)$ for all $x \in (0,+\infty)$. Equivalently,

$$F(u) \leq H(u) \leq G(u), \quad \text{for all} \quad u \in (0,+\infty).$$

Then, by Theorem 2, the functions F and G defined on $(0,+\infty)$ satisfy

$$F(tu + (1-t)v) \leq tG(u) + (1-t)G(v), \quad \text{for all } u, v \in (0,+\infty), \quad t \in [0,1].$$

For $x, y \in (0,+\infty)$ and $t \in [0,1]$,

$$\begin{aligned} f\left(\frac{xy}{tx + (1-t)y}\right) &= F\left(t\frac{1}{y} + (1-t)\frac{1}{x}\right) \\ &\leq tG\left(\frac{1}{y}\right) + (1-t)G\left(\frac{1}{x}\right) \\ &= tg(y) + (1-t)g(x). \end{aligned}$$

[(ii)⇒(i)] Conversely, if the inequalities (3) hold for all $x, y \in (0,+\infty)$ and $t \in [0,1]$, we consider the functions

$$F(x) := f\left(\frac{1}{x}\right) \qquad \text{and} \qquad G(x) := g\left(\frac{1}{x}\right).$$

For all $x, y \in (0,+\infty)$

$$\begin{aligned} f\left(\frac{xy}{tx + (1-t)y}\right) &\leq tg(y) + (1-t)g(x) \\ F\left(t\frac{1}{y} + (1-t)\frac{1}{x}\right) &\leq tG\left(\frac{1}{y}\right) + (1-t)G\left(\frac{1}{x}\right). \end{aligned}$$

Equivalently,

$$F(tv + (1-t)u) \leq tG(v) + (1-t)G(u), \quad \text{for all} \quad u, v \in (0,+\infty).$$

By Theorem 2, there is a convex function $H : (0,+\infty) \to \mathbb{R}$ such that $F \leq H \leq G$.

Now, if $x \in (0,+\infty)$,

$$\begin{array}{ccccc} F\left(\dfrac{1}{x}\right) & \leq & H\left(\dfrac{1}{x}\right) & \leq & G\left(\dfrac{1}{x}\right) \\ f(x) & \leq & h(x) & \leq & g(x), \end{array}$$

where $h : (0, +\infty) \to \mathbb{R}$, $h(x) := H\left(\dfrac{1}{x}\right)$. Note that the function h is convex, by virtue of Theorem 1. □

As a consequence of Theorem 3 we have the following.

Corollary 4. *If f, g_1, g_2 are real functions defined on the interval $(0, +\infty)$ and satisfy the inequality*

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq tg_1(y) + (1-t)g_2(x) \tag{4}$$

for all $x, y \in (0, +\infty)$ and $t \in [0, 1]$, then there exists a harmonically convex function $h : (0, +\infty) \to \mathbb{R}$ such that

$$f(x) \leq h(x) \leq \max\{g_1, g_2\}(x) \tag{5}$$

for all $x \in (0, +\infty)$.

Note that the reciprocal of this corollary does not hold. This can be verified easily by making use of the functions $h(x) = \ln_3(x)$, $f(x) = -\dfrac{1}{x} + 1$, $g_1(x) = 2$ and $g_2(x) = \dfrac{5}{9}x - 3$. Note that these functions satisfy the inequality (5), the particular values $t = \dfrac{1}{2}$, $x = 2$ and $y = 3$ do not satisfy the inequality (4).

Lemma 1. *If f is a harmonically convex function, then the function $\varphi = kf + \epsilon$ is also harmonically convex, for any constants ϵ and $k \in \mathbb{R}^+$.*

Proof. In fact,

$$\begin{aligned} \varphi\left(\frac{xy}{tx + (1-t)y}\right) &= kf\left(\frac{xy}{tx + (1-t)y}\right) + \epsilon \\ &\leq k(tf(y) + (1-t)f(x)) + \epsilon \\ &= ktf(y) + t\epsilon + (1-t)kf(x) + (1-t)\epsilon \\ &= t(kf(y) + \epsilon) + (1-t)(kf(x) + \epsilon) \\ &= t\varphi(y) + (1-t)\varphi(x). \end{aligned}$$

□

The next theorem, Theorem 6, is the second main result of this work, and it is about approximate convexity.

The Hyers-Ulam kind of stability problems of functional equations was originated by ULAM in 1940, when he proposed the following question [26]: Let f be a mapping from a group G_1 to a metric group G_2 with metric $d(\cdot,\cdot)$ such that

$$d(f(xy), f(x)f(y)) \leq \epsilon, \qquad x, y \in G_1.$$

Does there exist a group homomorphism h and $\delta_\epsilon > 0$ such that $d(f(x), h(x)) \leq \delta_\epsilon$, $x \in G_1$?.

One of the first assertions to be obtained is the following result, essentially due to HYERS [12], which gives an answer to ULAM's question.

Theorem 5. *Suppose that S is an additive semigroup, Y is a Banach space, $\epsilon \geq 0$, and $f : S \to Y$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon, \quad \text{for all} \quad x, y \in S. \tag{6}$$

Then there exists a unique function $A : S \to Y$ satisfying $A(x+y) = A(x) + A(y)$ and for which $\|f(x) - A(x)\| \leq \epsilon$ for all $x \in S$.

Since then, stability problems have been investigated in various directions for many other functional equations [17].

The investigation of approximate convexity probably started with the paper by HYERS and ULAM [13], who in 1952 introduced and investigated ϵ-convex functions: if D is a convex subset of a real linear space X and ϵ is a nonnegative number, then a function $f : D \to \mathbb{R}$ is called ϵ-convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \epsilon, \qquad x, y \in D, t \in [0,1].$$

HYERS and ULAM [13] proved that any $\epsilon-$convex function (where ϵ is a nonnegative number) on a finite dimensional convex set can be approximated by a convex function.

As an immediate consequence of Theorem 3 we obtain the following stability result of Hyers-Ulam type for harmonically convex functions (see [20, 21]).

Theorem 6. *Let $[a,b] \subseteq (0,+\infty)$ be an interval and $\epsilon > 0$. A function $f : [a,b] \to \mathbb{R}$ satisfies the inequality*

$$\left| f\left(\frac{xy}{tx + (1-t)y}\right) - tf(y) - (1-t)f(x) \right| \leq \epsilon, \tag{7}$$

for all $x, y \in [a,b]$ and $t \in [0,1]$ iff there exists an harmonically convex functions $\varphi : [a,b] \to \mathbb{R}$ such that

$$|f(x) - \varphi(x)| \leq \frac{\epsilon}{2}, \qquad x \in [a,b].$$

Proof. Define the function $g : [a, b] \to \mathbb{R}$ by $g(x) := f(x) + \epsilon$ then Theorem 3 holds with $g = f + \epsilon$, and it follows that there exists a harmonically convex function $h : [a, b] \to \mathbb{R}$ such that

$$f(x) \le h(x) \le f(x) + \epsilon,$$

for $x \in [a, b]$.

Putting $\varphi : [a, b] \to \mathbb{R}$ defined by $\varphi(x) := h(x) - \frac{\epsilon}{2}$, we obtain a harmonically convex function such that

$$\begin{aligned} f(x) - \frac{\epsilon}{2} \quad &\le \quad h(x) - \frac{\epsilon}{2} \quad \le \quad f(x) + \frac{\epsilon}{2} \\ -\frac{\epsilon}{2} \quad &\le \quad \varphi(x) - f(x) \quad \le \quad \frac{\epsilon}{2} \\ &\quad\ |\varphi(x) - f(x)| \quad \le \quad \frac{\epsilon}{2}, \end{aligned}$$

for all $x \in [a, b]$. □

We will now need the following setting: given $T > 0$ and $f : (0, +\infty) \to \mathbb{R}$, we define the function $f_T : (0, +\infty) \to \mathbb{R}$ by $f_T(x) := \frac{1}{T} f\left(\frac{1}{T} x\right)$.

The next theorem follows from the application of Theorem 3 concerning the solutions of the inequality

$$f\left(\frac{xy}{tx + (T - t)y}\right) \le tf(y) + (T - t)f(x). \tag{8}$$

Theorem 7. *Let T be a positive real number. A function $f : (0, +\infty) \to \mathbb{R}$ satisfies* (8) *for all $x, y \in (0, +\infty)$ and $t \in [0, T]$ if and only if exist a harmonically convex function $\varphi : I \subseteq (0, +\infty) \to \mathbb{R}$ such that*

$$\varphi_T \le f \le \varphi. \tag{9}$$

Proof. Assume that $f : (0, +\infty) \to \mathbb{R}$ satisfies (8) for any $x, y \in (0, +\infty)$ and $t \in [0, T]$. We can choose $\lambda \in [0, 1]$ such that $t = \lambda T$. Substituting λT for t in (8) we have

$$\begin{aligned} f\left(\frac{xy}{\lambda T x + (T - \lambda T)y}\right) &\le T \cdot \lambda f(y) + (T - \lambda T)f(x) \\ f\left(\frac{xy}{T(\lambda x + (1 - \lambda)y)}\right) &\le T\left(\lambda f(y) + (1 - \lambda)f(x)\right) \\ f_T\left(\frac{xy}{\lambda x + (1 - \lambda)y}\right) &\le \lambda f(y) + (1 - \lambda)f(x), \end{aligned} \tag{10}$$

for all $x, y \in (0, +\infty)$ and $\lambda \in [0, 1]$.

By Theorem 3, there exists a harmonically convex function $h : (0, +\infty) \to \mathbb{R}$ such that

$$f_T \leq h \leq f. \tag{11}$$

Define now $\varphi : (0, +\infty) \to \mathbb{R}$ by $\varphi(x) = Th\,(Tx)$. Note that

$$\begin{aligned}
\varphi\left(\frac{xy}{tx+(1-t)y}\right) &= Th\left(T\cdot\frac{xy}{tx+(1-t)y}\right) \\
&= Th\left(\frac{(Tx)(Ty)}{t(Tx)+(1-t)(Ty)}\right) \\
&\leq tTh(Ty)+(1-t)Th(Tx) \\
&= t\varphi(y)+(1-t)\varphi(x).
\end{aligned}$$

That is, φ is a harmonically convex function, and moreover

$$\begin{aligned}
\varphi_T(x) &= \frac{1}{T}\varphi\left(\frac{x}{T}\right) = \frac{1}{T}Th\left(T\frac{x}{T}\right) = h(x) \\
&\leq f(x) = T\cdot\frac{1}{T}f\left(\frac{1}{T}xT\right) = Tf_T(Tx) \\
&\leq Th(Tx) = \varphi(x).
\end{aligned}$$

Conversely, if there exists a harmonically convex function $\varphi : I \to \mathbb{R}$ that satisfies the inequality (9), we define now $\varphi(x) = Th\,(Tx)$. Then,

$$\begin{aligned}
h\left(\frac{xy}{tx+(1-t)y}\right) &= \frac{1}{T}\varphi\left(\frac{1}{T}\cdot\frac{xy}{tx+(1-t)y}\right) \\
&= \frac{1}{T}\varphi\left(\frac{\frac{x}{T}\frac{y}{T}}{t\frac{x}{T}+(1-t)\frac{y}{T}}\right) \\
&\leq \frac{1}{T}\left(t\varphi\left(\frac{y}{T}\right)+(1-t)\varphi\left(\frac{x}{T}\right)\right) \\
&= \frac{1}{T}\left(tTh\left(T\frac{y}{T}\right)+(1-t)Th\left(T\frac{x}{T}\right)\right) \\
&= th\,(y)+(1-t)h\,(x)\,.
\end{aligned}$$

That is, h is a harmonically convex function. On the other hand, we have

$$\begin{aligned}
f_T(x) &= \frac{1}{T}f\left(\frac{1}{T}x\right) \leq \frac{1}{T}\varphi\left(\frac{1}{T}x\right) = \frac{1}{T}\cdot T\cdot h\,(x) \\
&= h\,(x) = h\left(T\frac{x}{T}\right) = \frac{1}{T}\varphi\left(\frac{x}{T}\right) = \varphi_T\,(x)
\end{aligned}$$

Then, by Theorem 3 we have that

$$f_T\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y)+(1-t)f(x),$$

for all $x, y \in (0, +\infty)$ and $t \in [0, 1]$. But this means that, for all $x, y \in (0, +\infty)$ and $t \in [0, T]$, we get (8). $\square$

Let I denote the real interval either $(0, \infty)$ or $[a, b] \subset \mathbb{R} - \{0\}$, with $a < b$. Let $f : I \to \mathbb{R}$ be a function. Using Theorem 3, we describe also solutions of the inequality

$$f\left(\frac{1}{t\frac{1}{x} + (T-t)\frac{1}{y} + (1-T)\frac{1}{z_0}}\right) \leq tf(x) + (T-t)f(y) + (1-T)f(z_0). \tag{12}$$

Fix a real interval I and a point $z_0 \in I$. For $T \in (0, 1)$ put

$$I_T^* := \left\{x \in I : T \cdot \frac{xz_0}{z_0 - (1-T)x} \in I\right\}.$$

Given a real function φ with the domain containing I_T^*, we define $\varphi_T^* : I \to \mathbb{R}$ by

$$\varphi_T^*(x) = \frac{1}{T}\left[\varphi\left(\frac{xz_0}{Tz_0 + (1-T)x}\right) - (1-T)\varphi(z_0)\right].$$

Note that

$$\begin{aligned}
\varphi_T^*(z_0) &= \frac{1}{T}\left[\varphi\left(\frac{z_0^2}{Tz_0 + (1-T)z_0}\right) - (1-T)\varphi(z_0)\right] \\
&= \frac{1}{T}\left[\varphi(z_0) - \varphi(z_0) + T\varphi(z_0)\right] = \varphi(z_0).
\end{aligned}$$

Lemma 2. *If h is a harmonically convex function, then the function $g : I_T^* \to \mathbb{R}$ defined by $g(x) := h\left(T \cdot \frac{z_0 x}{z_0 - (1-T)x}\right)$ is harmonically convex.*

Proof. Let $x, y \in I_T^*$ and $t \in [0, 1]$, then

$$\begin{aligned}
g\left(\frac{xy}{ty + (1-t)x}\right) &= h\left(\frac{T \cdot z_0 \cdot \frac{xy}{ty + (1-t)x}}{z_0 - (1-T)\frac{xy}{ty + (1-t)x}}\right) \\
&= h\left(\frac{Tz_0xy}{z_0(ty + (1-t)x) - (1-T)xy}\right) \\
&= h\left(\frac{Tz_0xy}{tyz_0 - t(1-T)xy + (1-t)xz_0 - (1-t)(1-T)xy}\right) \\
&= h\left(\frac{T^2z_0^2xy}{Tz_0\{ty(z_0 - (1-T)x) + (1-t)x(z_0 - (1-T)y)\}}\right) \\
&= h\left(\frac{\frac{T^2z_0^2xy}{(z_0 - (1-T)y)(z_0 - (1-T)x)}}{\frac{Ttz_0y(z_0 - (1-T)x) + (1-t)Tz_0x(z_0 - (1-T)y)}{(z_0 - (1-T)y)(z_0 - (1-T)x)}}\right)
\end{aligned}$$

$$
\begin{aligned}
&= h\left(\frac{\dfrac{Tz_0y}{z_0-(1-T)y}\cdot\dfrac{Tz_0x}{z_0-(1-T)x}}{t\,\dfrac{Tz_0y}{z_0-(1-T)y}+(1-t)\,\dfrac{Tz_0x}{z_0-(1-T)x}}\right)\\
&\le th\left(\frac{Tz_0x}{z_0-(1-T)x}\right)+(1-t)h\left(\frac{Tz_0y}{z_0-(1-T)y}\right)\\
&= tg(x)+(1-t)g(y).
\end{aligned}
$$

□

Lemma 3. *If f satisfies the inequality*

$$f_T^*\left(\frac{xy}{\lambda y+(1-\lambda)x}\right) \le \lambda f(x)+(1-\lambda)f(y),$$

for all $x,y\in I$ and $\lambda\in[0,1]$, then f satisfies (12) *for all $x,y\in I$ and $\lambda\in[0,T]$.*

Proof. Let $x,y\in I$ and $t\in[0,1]$, we obtain

$$
\begin{aligned}
f_T^*\left(\frac{xy}{ty+(1-t)x}\right) &\le tf(x)+(1-t)f(y)\\
\frac{1}{T}\cdot\left[f\left(\frac{\dfrac{xy}{ty+(1-t)x}z_0}{Tz_0+(1-T)\dfrac{xy}{ty+(1-t)x}}\right)-(1-T)f(z_0)\right] &\le tf(x)+(1-t)f(y)\\
f\left(\frac{1}{T\left[t\dfrac{1}{x}+(1-t)\dfrac{1}{y}\right]+(1-T)\dfrac{1}{z_0}}\right) &\le T[tf(x)+(1-t)f(y)]+(1-T)f(z_0)\\
f\left(\frac{1}{Tt\dfrac{1}{x}+(T-Tt)\dfrac{1}{y}+(1-T)\dfrac{1}{z_0}}\right) &\le Ttf(x)+(T-Tt)f(y)+(1-T)f(z_0)
\end{aligned}
$$

Thus,

$$f\left(\frac{1}{\lambda\dfrac{1}{x}+(T-\lambda)\dfrac{1}{y}+(1-T)\dfrac{1}{z_0}}\right) \le \lambda f(x)+(T-\lambda)f(y)+(1-T)f(z_0)$$

for all $x,y\in I$ and $\lambda\in[0,T]$. □

Theorem 8. *Let $T\in(0,1)$. A function $f:I\to\mathbb{R}$ satisfies (12) for all $x,y\in I$ and $t\in[0,T]$ if only if there exists a harmonically convex function $\varphi:I_T^*\to\mathbb{R}$ such that*

$$\varphi_T^*(x)\le f(x),\ for\ x\in I \quad and\ \ f(x)\le\varphi(x),\quad for\ x\in I_T^*. \tag{13}$$

Proof. Assume that f satisfies (12) for any $x,y\in I$ and $t\in[0,T]$. We can choose $\lambda\in[0,1]$ such that $t=\lambda T$. Putting λT in place of t in (12) we get

$$\frac{1}{T}\left[f\left(\frac{\frac{xy}{\lambda y+(1-\lambda)x}z_0}{Tz_0+(1-T)\frac{xy}{\lambda y+(1-\lambda)x}}\right)-(1-T)f(z_0)\right] \quad \leq \quad \lambda f(x)+(1-\lambda)f(y).$$

Thus,

$$f^*_T\left(\frac{xy}{\lambda y+(1-\lambda)x}\right) \quad \leq \quad \lambda f(x)+(1-\lambda)f(y), \tag{14}$$

for all $x, y \in I$ and $\lambda \in [0,1]$.

Applying Theorem 3, we obtain a harmonically convex function $h : I \to \mathbb{R}$ such that

$$f^*_T(x) \leq h(x) \leq f(x) \tag{15}$$

for all $x \in I$. Since $f^*_T(z_0) = f(z_0)$,we have $h(z_0) = f(z_0)$. Define $\varphi : I^*_T \to \mathbb{R}$ by

$$\varphi(x) := T \cdot h\left(T \cdot \frac{z_0 x}{z_0-(1-T)x}\right)+(1-T)f(z_0). \tag{16}$$

By lemmas 1 and 2, we get that φ is a harmonically convex function. In addition, we will have that $\varphi(z_0) = f(z_0)$ and

$$\begin{aligned}
\varphi^*_T(x) &= \frac{1}{T}\left[\varphi\left(\frac{xz_0}{Tz_0+(1-T)x}\right)-(1-T)\varphi(z_0)\right] \\
&= \frac{1}{T}\left[Th\left(T\cdot\frac{z_0\frac{xz_0}{Tz_0+(1-T)x}}{z_0-(1-T)\frac{xz_0}{Tz_0+(1-T)x}}\right)+(1-T)f(z_0)-(1-T)\varphi(z_0)\right] \\
&= h\left(T\cdot\frac{\frac{xz_0^2}{Tz_0+(1-T)x}}{\frac{Tz_0^2+(1-T)xz_0-(1-T)xz_0}{Tz_0+(1-T)x}}\right) \\
&= h(x) \leq f(x),\ x \in I.
\end{aligned}$$

On the other hand, for all $x \in I^*_T$, we have

$$\begin{aligned}
\varphi(x) &= Th\left(T\cdot\frac{z_0x}{z_0-(1-T)x}\right)+(1-T)f(z_0) \\
&\geq Tf^*_T\left(T\cdot\frac{z_0x}{z_0-(1-T)x}\right)+(1-T)f(z_0) \\
&= T\left[\frac{1}{T}\left(f\left(\frac{T\cdot\frac{z_0x}{z_0-(1-T)x}z_0}{Tz_0+(1-T)T\cdot\frac{z_0x}{z_0-(1-T)x}}\right)-(1-T)f(z_0)\right)\right]+(1-T)f(z_0) \\
&= f\left(\frac{\frac{Tz_0^2x}{z_0-(1-T)x}}{\frac{Tz_0^2-T(1-T)xz_0+(1-T)Tz_0x}{z_0-(1-T)x}}\right)=f(x).
\end{aligned}$$

Conversely, if (13) holds with a harmonically convex function $\varphi : I^*_T \to \mathbb{R}$ then $f(z_0) = \varphi(z_0)$, and (16) defines a harmonically convex function $h : I \to \mathbb{R}$ which satisfies

$$h(x) = \frac{1}{T}\left[\varphi\left(\frac{xz_0}{Tz_0+(1-T)x}\right) - (1-T)f(z_0)\right]. \tag{17}$$

Thus, for any $x \in I$,

$$\begin{aligned} f_T^*(x) &= \frac{1}{T}\left[f\left(\frac{xz_0}{Tz_0+(1-T)x}\right) - (1-T)f(z_0)\right] \\ &\leq \frac{1}{T}\left[\varphi\left(\frac{xz_0}{Tz_0+(1-T)x}\right) - (1-T)f(z_0)\right] \\ &= h(x) \\ &= \frac{1}{T}\left[\varphi\left(\frac{xz_0}{Tz_0+(1-T)x}\right) - (1-T)\varphi(z_0)\right] \\ &= \varphi_T^*(x) \leq f(x). \end{aligned}$$

By Theorem 3 we obtain (14) for all $x, y \in I$ and $t \in [0, 1]$. By Lemma 3, f satisfies (12) for all $x, y \in I$ and $t \in [0, T]$. $\square$

3. Comments

In this paper we have two main results: a sandwich theorem for harmonically convex function and an approximate convexity result, namely, we proved a stability result of Hyers-Ulam type for harmonically convex functions. We expect that the ideas and techniques used in this paper may inspire interested readers to explore some new applications of these functions in various fields of pure and applied sciences.

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