Revista Colombiana de Matemáticas
Volumen 50(2016)2, páginas 143-162

New Hermite-Hadamard and Jensen Type Inequalities for $h-$Convex Functions on Fractal Sets

Nuevas desigualdades del tipo Hermite-Hadamard y Jensen para *funciones h-convexas* sobre conjuntos fractales

Miguel Vivas[1, 2,✉], Jorge Hernández[2], Nelson Merentes[3]

[1]Escuela Superior politécnica del Litoral (ESPOL), Guayaquil, Ecuador

[2]Universidad Centroccidental Lisandro Alvarado, Barquisimeto, Venezuela

[3]Universidad Central de Venezuela, Caracas, Venezuela

Abstract. In this paper, some new Jensen and Hermite-Hadamard inequalities for *h-convex functions* on fractal sets are obtained. Results proved in this paper may stimulate further research in this area.

Key words and phrases. generalized convexity, *h-convex functions*, Fractal sets, Hermite-Hadamard type inequality, Jensen inequality.

2010 Mathematics Subject Classification. 53C21, 53C42.

Resumen. En este artículo, se obtienen algunas nuevas desigualdades del tipo Jensen y Hermite - Hadamard para *funciones h-convexas* sobre conjuntos fractales. Los resultados probados en este artículo pueden estimular futuras investigaciones en esta área.

Palabras y frases clave. convexidad generalizada, *funciones h-convexas*, conjuntos fractales, desigualdad del tipo Hermite Hadamard, Desigualdad del tipo Jensen.

1. Introduction

Fractals have been known for about more than a century and have been observed in different branches of science. But it is only recently (approximately in the last forthy years) that they have become a subject of mathematical study. The pioneer of the theory of fractals was Benoit Mandelbrot. His book *Fractals: Form, Chance and Dimension* first appeared in 1977, and a second, enlarged, edition was published in 1982. Since that time, serious articles, surveys, popular papers, and books about fractals have appeared by the dozen. Mandelbrot in [17] defined a fractal set is one whose Hausdorff dimension exceeds strictly its topological dimension. Also, Yang in [30] established the numerical $\alpha-$sets, where α is the dimension of the considered fractal. For more details about fractal sets see for instance [6, 7, 8, 30] and references therein.

It is well known that modern analysis directly or indirectly involves the applications of convexity. Due to its applications and significant importance, the concept of convexity has been extended and generalized in several directions. The concept of convexity and its variant forms have played a fundamental role in the development of various fields. Convex functions are powerful tools for proving a large class of inequalities. They provide an elegant and unified treatment of the most important classical inequalities. A significant generalization of convex functions is that of *h-convex functions* introduced by Sanja Varošanec in [28]. There are many results associated with convex functions in the area of inequalities, two of those are: the Jensen inequality and the Hermite-Hadamard inequality, which occur widely in the mathematical literature. In this paper, we will establish some new integral inequalities of Hermite-Hadamard type for *h-convex functions*.

The following definition is well known in the literature as convex function: a function $f : I \subset R \to R$ is said to be convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.
The convexity of functions and their generalized forms play an important role in many fields such as Economic Science, Biology, Optimization. In recent years, several extensions, refinements, and generalizations have been considered for classical convexity [2, 5, 4, 16, 18, 19, 20, 26, 28].

The classical Jensen inequality is contained in the following theorem.

Theorem 1.1 (See [11]). *Let $f : I \subset R \to R$ be a convex function over I. Then for every $x_i \in I$, $t_i \in [0, 1]$, $i = 1, 2, \ldots, n$, and $\sum_{i=1}^{n} t_i = 1$, we have*

$$f\left(\sum_{i=1}^{n} t_i x_i\right) \leq \sum_{i=1}^{n} t_i f(x_i). \tag{1}$$

Jensen's inequality is sometimes called the king of inequalities since it implies the whole series of other classical inequalities (e.g. those by Hölder, Minkowski, Beckenbach-Dresher and Young, the arithmetic-geometric mean inequality etc.). Jensen's inequality for convex functions is probably one of the most important inequalities which is extensively used in almost all areas of mathematics, especially in mathematical analysis and statistics. For a comprehensive inspection of the classical and recent results related to the inequality (1) the reader is referred to [20, 25, 27, 29].

One of the goals of this article is to establish a Jensen-type inequality for generalized *h-convex functions.*

It is well-known that one of the most fundamental and interesting inequalities for classical convex functions is that associated with the name of Hermite-Hadamard inequality which provides a lower and an upper estimations for the integral average of any convex functions defined on a compact interval, involving the midpoint and the endpoints of the domain. More precisely:

Theorem 1.2 (See [10])**.** *Let f be a convex function over $[a,b]$, $a<b$. If f is integrable over $[a,b]$, then*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{(b-a)}\int_a^b f(x)\,dx\leq\frac{f(a)+f(b)}{2}. \tag{2}$$

The above inequality (2) was firstly discovered by Hermite in 1881 in the journal Mathesis (see Mitrinović and Lačković [19]). But, this beautiful result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result (see Klaričić et al. [1]). For more recent results which generalize, improve, and extend the classical Hermite-Hadamard inequality (2), see for instance [15, 24, 25], and references therein. The Hermite-Hadamard inequality has several applications in nonlinear analysis and the geometry of Banach spaces, see [12, 23].

In the present paper,we are concerned with an analogous of Theorem 2 for *h-convex functions* on fractal set. Let us recall two important definitions of generalized convex functions.

Definition 1.3 (See [9])**.** We shall say that a function $f: I\subset R\to R$ is a *Godunova-Levin* function or $f\in Q(I)$ if f is non negative and for each $x,y\in I$ and $t\in(0,1)$ we have

$$f(tx+(1-t)y)\leq\frac{f(x)}{t}+\frac{f(y)}{1-t}.$$

Definition 1.4 (See [3])**.** Let $s\in(0,1]$. A function $f:(0,\infty]\to(0,\infty]$ is called a $s-convex$ function (in the second sense), or $f\in K_s^2$ if

$$f(tx+(1-t)y)\leq t^s f(x)+(1-t)^s f(y)$$

for each $x,y\in(0,\infty]$ and $t\in[0,1]$.

It is clear that,for $s = 1$, $s-$convexity reduces to ordinary convexity of functions defined on $(0, \infty]$.

In the year 1999, Dragomir [5] proved a variant of the Hermite-Hadamard inequality (2), for s-convex functions in the second sense.

Theorem 1.5 (See [5])**.** *Let $f : (0, \infty] \to (0, \infty]$ a $s-$convex function in the second sense,with $s \in (0, 1]$, and $a, b \in (0, \infty]$, $a < b$. If $f \in L^1([a, b])$, then we have*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{(b-a)} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{s+1}.$$

In the year 2007, Varošanec, [28], defined the following so-called *h-convex function*:

Definition 1.6. Let $h : J \to R$ be a non-negative,non-identically zero function, defined on an interval $J \subset R$, with $(0, 1) \subset J$. We shall say that a function $f : I \to R$, defined on an interval $I \subset R$, is *h-convex* if f is non negative and this inequality holds

$$f\left(tx + (1-t)\, y\right) \leq h\left(t\right) f\left(x\right) + h\left(1-t\right) f\left(y\right)$$

for all $t \in (0, 1)$ and $x, y \in I$.

When $h(t) = t$, this definition coincides with the ordinary convex function. If $h(t) = t^s$ with $0 < s \leq 1$, the coincidence is with the $s-$convex functions, and if $h(t) = 1/t$ this coincides with the Godunova-Levin type of generalized convex function.

For other recent results and properties of the class of *h-convex functions* see [2, 16, 13, 14, 22].

In this article motivated and inspired by the ongoing research in the field [13, 14, 21, 22],we establish new Hermite-Hadamard and Jensen type inequalities for *h-convex functions* on fractal sets.

The article is organized as follows: In section 2 we state the operations with real line numbers on fractal sets and we recall some definitions and preliminary facts of fractional calculus theory which will be used in this paper, also we introduce the definition of *h-convexity* on fractal sets. In secction 3,we establish the main results of the article: the generalized Jense‘s inequality and generalized Hermite-Hadamard‘s inequalty for generalized *h-convex functions*. In section 4 we give some applications/examples to illustrate.

2. Preliminaries and Basic Results

Recently, the theory of Yang's fractional set of elements sets was introduced as follows:

For $0 < \alpha \leq 1$ we have the following $\alpha-$type sets.

- $Z^\alpha = \{0^\alpha, \pm 1^\alpha, \pm 2^\alpha, ..., \pm n^\alpha, ...\}$
- $Q^\alpha = \{(a/b)^\alpha : a^\alpha \in Z^\alpha, b^\alpha \in Z^\alpha, b^\alpha \neq 0^\alpha\}$
- $I^\alpha = \{m^\alpha \neq (a/b)^\alpha : a^\alpha \in Z^\alpha, b^\alpha \in Z^\alpha, b^\alpha \neq 0^\alpha\}$
- $R^\alpha = Q^\alpha \cup I^\alpha$

For $a^\alpha, b^\alpha, c^\alpha \in R^\alpha$ the following properties hold:

a. $a^\alpha + b^\alpha \in R^\alpha$ y $a^\alpha b^\alpha \in R^\alpha$

b. $a^\alpha + b^\alpha = b^\alpha + a^\alpha = (a+b)^\alpha = (b+a)^\alpha$

c. $a^\alpha + (b^\alpha + c^\alpha) = (a^\alpha + b^\alpha) + c^\alpha$

d. $a^\alpha b^\alpha = b^\alpha a^\alpha = (ab)^\alpha = (ba)^\alpha$

e. $a^\alpha (b^\alpha c^\alpha) = (a^\alpha b^\alpha) c^\alpha$

f. $a^\alpha + 0^\alpha = 0^\alpha + a^\alpha = a^\alpha$ y $a^\alpha 1^\alpha = 1^\alpha a^\alpha = a^\alpha$

If $a^\alpha - b^\alpha$ is non negative we say a^α is greater than or equal to b^α , or b^α is less than or equal to a^α, and we write $a^\alpha \geq b^\alpha$ or $b^\alpha \leq a^\alpha$, respectively. If there is not possibility that $a^\alpha = b^\alpha$ then we write $a^\alpha > b^\alpha$ o $b^\alpha < a^\alpha$.

Next we recall some definitions and some facts of fractional calculus theory on R^α which will be used in this paper.

Definition 2.1. Let $f : R \to R^\alpha$ be a mapping. We say that f is local fractional continuous at $x_0 \in R$, if for all $\epsilon > 0$ exists $\delta > 0$ such that

$$|x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| < \epsilon^\alpha$$

If f is local fractional continuous in each point of an interval (a, b), we say that f is local fractional continuous in (a, b) and we write $f \in C_\alpha(a, b)$.

Definition 2.2. The local fractional derivative of f of order α at $x = x_0$ is defined by

$$f^{(\alpha)}(x_0) = \left.\frac{d^\alpha f(x)}{dx^\alpha}\right|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^\alpha (f(x) - f(x_0))}{(x - x_0)^\alpha}$$

where $\Delta (f(x) - f(x_0)) \cong \Gamma(\alpha + 1)(f(x) - f(x_0))$.

Definition 2.3. Let $f \in C_\alpha[a, b]$. Then the local fractional integral of order α of f is defined by

$$\begin{aligned} {}_aI_b^{(\alpha)} f =& \frac{1}{\Gamma(1+\alpha)} \int_a^b f(x) (dx)^\alpha \\ =& \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{i=1}^{N} f(t_i) (\Delta t_i)^\alpha \end{aligned}$$

where $\Delta t_i = t_{i+1} - t_i$, $\Delta t = max\{\Delta t_1, \dots, \Delta t_N\}$, and $[t_i, t_{i+1}]$, $i = 1, 2, \dots, N$, with $a = t_0 < t_1 < \cdots < t_{N-1} = b$, is a partition of $[a, b]$.

If for each $x \in [a, b]$ there exists ${}_aI_b^{(\alpha)}f$, then we write $f \in I_x^{(\alpha)}[a, b]$.

Here, it follows

$$ {}_aI_b^{(\alpha)}f = 0 \text{ if } a = b $$

and

$$ {}_bI_a^{(\alpha)}f = -{}_aI_b^{(\alpha)}f \text{ if } a < b. $$

Also we have the property of change of variables.

Lemma 2.4. *If $g \in C_\alpha[a, b]$ and $f \in C_\alpha[g(a), g(b)]$ then*

$$ \begin{aligned} {}_{g(a)}I_{g(b)}^{(\alpha)}f &= \frac{1}{\Gamma(1+\alpha)} \int_{g(a)}^{g(b)} f(x)(dx)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \int_a^b f(g(t))\, g'(t)(dt)^\alpha \\ &= {}_a I_b^{(\alpha)}((f \circ g)\, g'). \end{aligned} $$

In [22], Mo and Sui considered the following denfition of generalized convexity on fractal set.

Definition 2.5. Let $f : I \to R^\alpha$, with $0 < \alpha \le 1$. For any $x_1 \neq x_2$ in I and $t \in [0, 1]$, we say that f is generalized convex function on I if

$$ f(tx_1 + (1-t)x_2) \le t^\alpha f(x_1) + (1-t)^\alpha f(x_2) $$

holds.

In [21], the definition of $s-$convex functions on fractal sets was established as follows:

Definition 2.6. A function $f : R_+ \to R^\alpha$ is said to be a generalized $s-$convex $(0 < s < 1)$ in the second sense, if

$$ f(t_1x_1 + t_2x_2) \le t_1^{s\alpha} f(x_1) + t_2^{s\alpha} f(x_2) $$

for all $x_1, x_2 \in R_+$ and all $t_1, t_2 > 0$ with $t_1 + t_2 = 1$.

With this, they obtain the following results.

Theorem 2.7. *Let $f : I \to R^\alpha$ be a generalized convex function. Then for each $x_i \in [a, b]$ and $t_i \in [0, 1]$ with $i = 1, 2, \dots, n$ we have*

$$ f\left(\sum_{i=1}^n t_i x_i\right) \le \sum_{i=1}^n t_i^\alpha f(x_i). $$

Theorem 2.8. *Let $f \in I_x^{(\alpha)}[a,b]$ be a generalized convex function on $[a,b]$ with $a<b$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\int_a^b f(x)(dx)^\alpha \leq \frac{f(a)+f(b)}{2^\alpha}.$$

Next we give our definition of generalized h *-convex functions* on fractal set.

Definition 2.9. Let $h : J \to R^\alpha$ be a non-negative function and $h \not\equiv 0$, defined over an interval $J \subset R$ and such that $(0,1) \subset J$. We say that $f : I \to R^\alpha$, defined over an interval $I \subset R$, is *h-convex* if f is non negative and we have

$$f\left(tx + (1-t)\,y\right) \leq h\,(t)\,f\,(x) + h\,(1-t)\,f\,(y) \tag{3}$$

for all $t \in (0,1)$ and $x, y \in I$.

We can see that if $h(t) \geq t^\alpha$, like $h(t) = t^{k\alpha}$, where $0 < k \leq 1$ then any non-negative and convex function $f : I \to R^\alpha$ is *h-convex* on R^α.

In [21] we can find another example of such functions.

Example 2.10. Let $0 < s < 1$, $h : (0,1) \to R^\alpha$ defined by $h(t) = t^{s\alpha}, (t \in (0,1))$ and $a^\alpha, b^\alpha, c^\alpha \in R^\alpha$. For $x \in R_+$, define

$$f(x) = \begin{cases} a^\alpha, & \text{si } \; x = 0 \\ b^\alpha x^{s\alpha} + c^\alpha, & \text{si } \; x > 0 \end{cases}$$

3. Main Results

In this section, we establish our main results.

Theorem 3.1. *Let $t_1, \ldots, t_n$ be positive real numbers. If $h : J \to R^\alpha$ is a non-negative function, $h \not\equiv 0$, supermultiplicative defined over an interval $J \subset R$ and such that $(0,1) \subset J$, and let $f : I \to R^\alpha$ be a function defined over an interval $I \subset R$, $h-$convex, and $x_1, \ldots, x_n \in I$, then*

$$f\left(\frac{1}{T_n}\sum_{i=1}^n t_i x_i\right) \leq \sum_{i=1}^n h\left(\frac{t_i}{T_n}\right) f\left(x_i\right) \tag{4}$$

where $T_n = \sum_{i=1}^n t_i$.

Proof. The proof is by induction. If $n = 2$, the desired inequality is obtained from the definition of *h-convex function* (3) with $t = \frac{t_1}{T_2}$, $(1-t) = \frac{t_2}{T_2}$, $x = x_1$ and $y = x_2$.

Assume that for $n-1$, where n is any positive integer, the inequality (4) is also true.

Then, we see that

$$f\left(\frac{1}{T_n}\sum_{i=1}^{n} t_i x_i\right) = f\left(\frac{t_n}{T_n}x_n + \frac{1}{T_n}\sum_{i=1}^{n-1} t_i x_i\right)$$
$$= f\left(\frac{t_n}{T_n}x_n + \frac{T_{n-1}}{T_n}\sum_{i=1}^{n-1} \frac{t_i}{T_{n-1}} x_i\right).$$

Using the definition (2.9) in the right-hand side of the previous inequality, we have

$$f\left(\frac{1}{T_n}\sum_{i=1}^{n} t_i x_i\right) \le h\left(\frac{t_n}{T_n}\right) f\left(x_n\right) + h\left(\frac{T_{n-1}}{T_n}\right) f\left(\sum_{i=1}^{n-1} \frac{t_i}{T_{n-1}} x_i\right).$$

Now, as we have assumed that (4) holds for $n-1$ we obtain

$$f\left(\frac{1}{T_n}\sum_{i=1}^{n} t_i x_i\right) \le h\left(\frac{t_n}{T_n}\right) f\left(x_n\right) + h\left(\frac{T_{n-1}}{T_n}\right) \sum_{i=1}^{n-1} h\left(\frac{t_i}{T_{n-1}}\right) f\left(x_i\right)$$
$$= h\left(\frac{t_n}{T_n}\right) f\left(x_n\right) + \sum_{i=1}^{n-1} h\left(\frac{T_{n-1}}{T_n}\right) h\left(\frac{t_i}{T_{n-1}}\right) f\left(x_i\right).$$

Further, since h is a supermultiplicative function, we can see

$$h\left(\frac{T_{n-1}}{T_n}\right) h\left(\frac{t_i}{T_{n-1}}\right) \le h\left(\frac{T_{n-1}}{T_n}\frac{t_i}{T_{n-1}}\right) = h\left(\frac{t_i}{T_n}\right),$$

using this fact we obtain

$$f\left(\frac{1}{T_n}\sum_{i=1}^{n} t_i x_i\right) \le h\left(\frac{t_n}{T_n}\right) f\left(x_n\right) + \sum_{i=1}^{n-1} h\left(\frac{t_i}{T_n}\right) f\left(x_i\right) = \sum_{i=1}^{n} h\left(\frac{t_i}{T_n}\right) f\left(x_i\right).$$

The above inequality holds by the result for n=2 and the induction hypothesis. ☑

***Remark* 3.2.** If $h(t) = t^{\alpha}$ we have

$$f\left(\frac{1}{T_n}\sum_{i=1}^{n} t_i x_i\right) \le \sum_{i=1}^{n} \left(\frac{t_i}{T_n}\right)^{\alpha} f\left(x_i\right)$$

and if we put $\lambda_i = \left(t_i/T_n\right), (i = 1, ..n)$ then

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \le \sum_{i=1}^{n} \lambda_i^{\alpha} f\left(x_i\right)$$

and this coincides with the result demonstrated by Mo and Sui in [22] about generalized convex function over fractal set. In the same way if $h(t) = t^{s\alpha}, (0 < s < 1)$, we have

$$f\left(\frac{1}{T_n}\sum_{i=1}^{n} t_i x_i\right) \leq \sum_{i=1}^{n}\left(\frac{t_i}{T_n}\right)^{s\alpha} f\left(x_i\right)$$

and if $\lambda_i = \left(t_i/T_n\right), (i = 1, ..n)$ then

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq \sum_{i=1}^{n} \lambda_i^{s\alpha} f\left(x_i\right)$$

corresponding to generalized $s-$convex functions over fractal sets.

The next result involves an integral inequality of *Hermite-Hadamard* type.

Theorem 3.3. *Let $h : J \to R^\alpha$ be a non-negative integrable function, $h \not\equiv 0$, defined over an interval $J \subset R$ and such that $(0,1) \subset J$ and $f : I \to R^\alpha$ be an $h-$ convex, non-negative and integrable function, $a, b \in I$ with $a < b$. Then*

$$\frac{1}{\left(1-(-1)^\alpha\right) h(1/2)\Gamma\left(1+\alpha\right)} f(\frac{a+b}{2}) \leq \frac{1}{(b-a)^\alpha}\ {}_aI_b^{(\alpha)} f \tag{5}$$
$$\leq \left(f\left(b\right) - (-1)^\alpha f\left(a\right)\right)\ {}_0I_1^{(\alpha)} h.$$

Proof. Note that

$$ta + (1-t)\,b + (1-t)\,a + tb = ta + b - tb + a - ta + tb = a + b$$

for all $t \in [0, 1]$. And as f is an *h-convex function*, we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq h\left(1/2\right) f\left(ta + (1-t)\,b\right) + h\left(1/2\right) f\left((1-t)\,a + tb\right) \\ &= h\left(1/2\right)\left(f\left(ta + (1-t)\,b\right) + f\left((1-t)\,a + tb\right)\right). \end{aligned}$$

Thus, integrating both sides, we get

$$\begin{aligned} &\int_0^1 f\left(\frac{a+b}{2}\right)(dt)^\alpha \\ &\quad\leq h\left(1/2\right)\int_0^1 f\left(ta + (1-t)\,b\right)(dt)^\alpha + h\left(1/2\right)\int_0^1 f\left((1-t)\,a + tb\right)(dt)^\alpha. \end{aligned}$$

Now, we note that

$$\int_0^1 f\left(ta + (1-t)\,b\right)(dt)^\alpha = \frac{-(-1)^\alpha}{(b-a)^\alpha}\int_a^b f(x)\,(dx)^\alpha$$

and

$$\int_0^1 f\left(\left(1-t\right)a+tb\right)\left(dt\right)^\alpha = \frac{1}{(b-a)^\alpha}\int_a^b f(x)\left(dx\right)^\alpha$$

and with this we have

$$\int_0^1 f\left(\frac{a+b}{2}\right)\left(dt\right)^\alpha \le \frac{h\left(1/2\right)}{(b-a)^\alpha}\left(1-(-1)^\alpha\right)\int_a^b f(x)\left(dx\right)^\alpha$$

from which it follows that

$$\frac{1^\alpha}{\left(1-(-1)^\alpha\right)h\left(1/2\right)\Gamma\left(1+\alpha\right)}f\left(\frac{a+b}{2}\right) \le \frac{1}{(b-a)^\alpha}\ {}_aI_b^{(\alpha)}f$$

which corresponds to the left inequality in (3.3).

We know that for any $x\in[a,b]$ there exists $t\in[0,1]$ such that $x=ta+\left(1-t\right)b$. With this fact and the *h-convexity* of f, we can write

$$\begin{aligned}
\int_a^b f\left(x\right)\left(dx\right)^\alpha &= (b-a)^\alpha \int_0^1 f\left(\left(1-t\right)a+tb\right)\left(dt\right)^\alpha \\
&\le (b-a)^\alpha \int_0^1 \left(h\left(1-t\right)f\left(a\right)+h\left(t\right)f\left(b\right)\right)\left(dt\right)^\alpha \\
&= (b-a)^\alpha\left(f\left(a\right)\int_0^1 h\left(1-t\right)\left(dt\right)^\alpha + f\left(b\right)\int_0^1 h\left(t\right)\left(dt\right)^\alpha\right) \\
&= (b-a)^\alpha\left(-f\left(a\right)\left(-1\right)^\alpha\int_0^1 h\left(t\right)\left(dt\right)^\alpha + f\left(b\right)\int_0^1 h\left(t\right)\left(dt\right)^\alpha\right) \\
&= (b-a)^\alpha\left(-(-1)^\alpha f\left(a\right)+f\left(b\right)\right)\int_0^1 h\left(t\right)\left(dt\right)^\alpha
\end{aligned}$$

an so we obtain

$$\frac{1}{(b-a)^\alpha}\ {}_aI_b^{(\alpha)}f \le \left(-(-1)^\alpha f\left(a\right)+f\left(b\right)\right)_0 I_1^{(\alpha)}h$$

which corresponds to the right-hand side of (3.3), and we can conclude

$$\begin{aligned}
\frac{1}{\left(1-(-1)^\alpha\right)h(1/2)\Gamma\left(1+\alpha\right)}f\left(\frac{a+b}{2}\right) &\le \frac{1}{(b-a)^\alpha}\ {}_aI_b^{(\alpha)}f \\
&\le \left(f\left(b\right)-(-1)^\alpha f\left(a\right)\right)_0 I_1^{(\alpha)}h.
\end{aligned}$$

This complete the proof. ☑

***Remark* 3.4.** Observe that if $h(t)=t$ and $\alpha=1$ then, $\Gamma(1+\alpha)=\Gamma(2)=1$, $h(1/2)=1/2$,

$${}_0I_1^{(\alpha)}h = \frac{1}{\Gamma\left(1+\alpha\right)}\int_0^1 h\left(t\right)\left(dt\right)^\alpha = \frac{1}{2}$$

and

$$_aI_b^{(\alpha)}f = \int_a^b f(x)dx$$

consequently we get (2) from (3.3).

***Remark* 3.5.** If $h(t) = t^s$ with $s \in (0,1]$ y $\alpha = 1$, then $\Gamma(1+\alpha) = \Gamma(2) = 1$ y $h(1/2) = 1/2^s$ we get

$$_0I_1^{(\alpha)}h = \frac{1}{\Gamma(1+\alpha)} \int_0^1 h(t)(dt)^\alpha = \frac{1}{s+1}$$

and

$$_aI_b^{(\alpha)}f = \int_a^b f(x)dx$$

therefore

$$2^{s-1}f(\frac{a+b}{2}) \le \frac{1}{(b-a)} \int_a^b f(x)dx \le \frac{f(b)+f(a)}{s+1}$$

and it corresponds to the result obtained in [5] for $s-$convex functions in the second sense.

Theorem 3.6. *Let $h_1, h_2 : J \to R^\alpha$ be two non-negative functions and $h_1, h_2 \not\equiv 0$, defined over an interval $J \subset R$ and such that $(0,1) \subset J$, moreover $h_1 \in I_x^{(\alpha)}[0,1], h_2 \in I_x^{(\alpha)}[0,1]$ and $(h_1h_2) \in I_x^{(\alpha)}[0,1]$. Let f be an h_1-convex function, and g an h_2- convex function, both non-negative on R^α , $a, b \in I$, $a < b$ and such that $(fg) \in I_x^{(\alpha)}[a,b]$. Then*

$$\frac{-(-1)^\alpha}{(b-a)^\alpha} {_aI_b^{(\alpha)}}(fg) \le M(a,b)_0I_1^{(\alpha)}(h_1h_2) + N(a,b)_0I_1^{(\alpha)}(h_1h_2(1-t)) \qquad (6)$$

where

$$M(a,b) = f(a)g(a) + (-1)^\alpha f(b)g(b) \text{ and } N(a,b) = f(a)g(b) + (-1)^\alpha f(b)g(a).$$

Proof. Since f is a $h_1 - convex function$, and g is a $h_2 - convex function$, and for each $x \in [a,b]$ exists $t \in [0,1]$ such that $x = ta + (1-t)b$, we have

$$f(ta + (1-t)b) \le h_1(t)f(a) + h_1(1-t)f(b)$$

and

$$g(ta + (1-t)b) \le h_2(t)g(a) + h_2(1-t)g(b).$$

Further, since f and g are non-negative, then

$$\begin{aligned} & f(ta + (1-t)b)g(ta + (1-t)b) \\ & \quad \le (h_1(t)f(a) + h_1(1-t)f(b))(h_2(t)g(a) + h_2(1-t)g(b)) \end{aligned}$$

$$= h_1(t)h_2(t)f(a)g(a) + h_1(t)h_2(1-t)f(a)g(b)$$
$$+h_1(1-t)h_2(t)f(b)g(a) + h_1(1-t)h_2(1-t)f(b)g(b)$$

integrating over $[0,1]$ both sides of the inequality, we obtain

$$\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)(dt)^\alpha$$

$$\begin{aligned}
&\leq f(a)g(a)\int_0^1 h_1(t)h_2(t)(dt)^\alpha \\
&+ f(a)g(b)\int_0^1 h_1(t)h_2(1-t)(dt)^\alpha \\
&+ f(b)g(a)\int_0^1 h_1(1-t)h_2(t)(dt)^\alpha \\
&+ f(b)g(b)\int_0^1 h_1(1-t)h_2(1-t)(dt)^\alpha \\
&= (f(a)g(a) - (-1)^\alpha f(b)g(b))\int_0^1 h_1(t)h_2(t)(dt)^\alpha \\
&+ (f(a)g(b) - (-1)^\alpha f(b)g(a))\int_0^1 h_1(t)h_2(1-t)(dt)^\alpha .
\end{aligned}$$

From the proof of the previous Theorem we have

$$\int_0^1 f\left(ta+(1-t)\,b\right)(dt)^\alpha = \frac{-(-1)^\alpha}{(b-a)^\alpha}\int_a^b f(x)\,(dx)^\alpha .$$

Then

$$\frac{-(-1)^\alpha}{(b-a)^\alpha}\int_a^b f(x)g(x)\,(dx)^\alpha \leq M(a,b)_0I_1^{(\alpha)}(h_1h_2) + N(a,b)_0I_1^{(\alpha)}(h_1h_2(1-\cdot))$$

where

$$M(a,b) = f(a)g(a) + (-1)^\alpha f(b)g(b)$$

and

$$N(a,b) = f(a)g(b) + (-1)^\alpha f(b)g(a).$$

☑

Theorem 3.7. *Let $h_1, h_2 : J \to R^\alpha$ be two non negative functions and $h_1, h_2 \not\equiv 0$, defined over an interval $J \subset R$ and such that $(0,1) \subset J$, and $(h_1h_2) \in I_x^{(\alpha)}[0,1]$ Let f an h_1-convex function, and g an h_2- convex function, both non-negative over R^α , $a, b \in I$, $a < b$ such that $(fg) \in I_x^{(\alpha)}[a,b]$. Then*

$$\frac{(1)^\alpha}{(1+(-1)^\alpha)h_1(1/2)h_2(1/2)}f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\ {}_aI_b^{(\alpha)}(fg)$$

$$\leq M(a,b)\Gamma(1+\alpha)_0I_1^{(\alpha)}(h_1h_2(1-\cdot)) + N(a,b)\Gamma(1+\alpha)_0I_1^{(\alpha)}(h_1h_2)$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$ *and* $N(a,b) = f(a)g(b) + f(b)g(a)$.

Proof. Let $a, b \in I$ with $a < b$. Then we can write

$$\frac{a+b}{2} = \frac{ta+(1-t)b}{2} + \frac{(1-t)a+tb}{2}$$

for all $t \in [0,1]$. In consequence,

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) = f\left(\frac{ta+(1-t)b}{2} + \frac{(1-t)a+tb}{2}\right) \times$$
$$g\left(\frac{ta+(1-t)b}{2} + \frac{(1-t)a+tb}{2}\right).$$

Since f is $h_1 - convex$ and g is $h_2 - convex$, we have

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq h_1(1/2)[f\,(ta+(1-t)b) + f\,((1-t)a+tb)]$$
$$\times\, h_2(1/2)[g\,(ta+(1-t)b) + g\,((1-t)a+tb)].$$

Using distributive property we get

$$\begin{aligned} f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq\; & h_1(1/2)h_2(1/2)[f\,(ta+(1-t)b)\,g\,(ta+(1-t)b) \\ & + f\,(ta+(1-t)b)\,g\,((1-t)a+tb) \\ & + f\,((1-t)a+tb)\,g\,(ta+(1-t)b) \\ & + f\,((1-t)a+tb)\,g\,((1-t)a+tb)]. \end{aligned}$$

Forming groups with the terms we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq\; & h_1(1/2)h_2(1/2)[f\,(ta+(1-t)b)\,g\,(ta+(1-t)b) \\ & +f\,((1-t)a+tb)\,g\,((1-t)a+tb)]+ \\ & h_1(1/2)h_2(1/2)[f\,(ta+(1-t)b)\,g\,((1-t)a+tb) \\ & +f\,((1-t)a+tb)\,g\,(ta+(1-t)b)]. \end{aligned} \tag{7}$$

Again, using the $h_1 - convexity$ and $h_2 - convexity$ of f and g respectively, and distributing the products in the second term of the sum in the previous inequality, we can observe

$$\begin{aligned} & f\,(ta+(1-t)b)g\,((1-t)a+tb) + f\,((1-t)a+tb)\,g\,(ta+(1-t)b) \\ & \leq h_1(t)f(a)h_2(1-t)g(a) + h_1(t)f(a)h_2(t)g(b) \\ & + h_1(1-t)f(b)h_2(1-t)g(a) + h_1(1-t)f(b)h_2(t)g(b) \\ & + h_1(1-t)f(a)h_2(t)g(a) + h_1(1-t)f(a)h_2(1-t)g(b) \\ & + h_1(t)f(b)h_2(t)g(a) + h_1(t)f(b)h_2(1-t)g(b). \end{aligned}$$

Now, we grouped the terms conveniently

$$\begin{aligned} f\left(ta+(1-t)b\right)g\left((1-t)a+tb\right)+f\left((1-t)a+tb\right)g\left(ta+(1-t)b\right) \\ =\{h_1(t)h_2(1-t)+h_1(1-t)h_2(t)\}M(a,b) \\ +\{h_1(t)h_2(t)+h_1(1-t)h_2(1-t)\}N(a,b) \end{aligned}$$

where

$$M(a,b)=f(a)g(a)+f(b)g(b)$$

and

$$N(a,b)=f(a)g(b)+f(b)g(a).$$

In consequence, the inequality (7) takes the form

$$\begin{aligned} f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq h_1(1/2)h_2(1/2)[f\left(ta+(1-t)b\right)g\left(ta+(1-t)b\right) \quad (8) \\ +f\left((1-t)a+tb\right)g\left((1-t)a+tb\right)] \\ +h_1(1/2)h_2(1/2)\left([h_1(t)h_2(1-t)+h_1(1-t)h_2(t)]\,M(a,b)\right. \\ \left.+\,[h_1(t)h_2(t)+h_1(1-t)h_2(1-t)]\,N(a,b)\right). \end{aligned}$$

Observe the following integrals

$$\int_0^1 f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)(dt)^\alpha=(1)^\alpha f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right),$$

$$\int_0^1 h_1(1-t)h_2(1-t)(dt)^\alpha=-(-1)^\alpha\int_0^1 h_1(t)h_2(t)(dt)^\alpha,$$

$$\int_0^1 h_1(1-t)h_2(t)(dt)^\alpha=-(-1)^\alpha\int_0^1 h_1(t)h_2(1-t)(dt)^\alpha$$

and making the substitution $x=at+(1-t)b$ we get

$$\int_0^1 \left(f\left(ta+(1-t)b\right)g\left(ta+(1-t)b\right)\right)(dt)^\alpha=\frac{-\left(-1\right)^\alpha}{(b-a)^\alpha}\int_a^b f(x)g(x)(dx)^\alpha$$

and with the substitution $x=(1-t)a+tb$ we have

$$\int_0^1 \left(f\left((1-t)a+tb\right)g\left((1-t)a+tb\right)\right)(dt)^\alpha=\frac{1}{(b-a)^\alpha}\int_a^b f(x)g(x)(dx)^\alpha.$$

So, with this changes and integrating both sides of the inequality (8) over $[0,1]$ we obtain

$$\begin{aligned}(1)^{\alpha} f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\ &\leq h_1(1/2)h_2(1/2)\left(1-(-1)^{\alpha}\right)\frac{1}{(b-a)^{\alpha}}\int_0^1 f(x)g(x)(dx)^{\alpha} \\ &+ h_1(1/2)h_2(1/2)\{M(a,b)\left(1-(-1)^{\alpha}\right)\int_0^1 h_1(t)h_2(1-t)(dt)^{\alpha} \\ &+ N(a,b)\left(1-(-1)^{\alpha}\right)\int_0^1 h_1(t)h_2(t)(dt)^{\alpha}\}\end{aligned}$$

and follows

$$\frac{(1)^{\alpha}}{(1+(-1)^{\alpha})h_1(1/2)h_2(1/2)} f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) - \frac{1}{(b-a)^{\alpha}}\int_a^b f(x)g(x)(dx)^{\alpha}$$

$$\leq M(a,b)\int_0^1 (h_1(t)h_2(1-t))(dt)^{\alpha} + N(a,b)\int_0^1 (h_1(t)h_2(t))(dt)^{\alpha}.$$

Using the fractional integral definition, this inequality can be written as

$$\frac{(1)^{\alpha}}{(1-(-1)^{\alpha})h_1(1/2)h_2(1/2)} f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) - \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{(\alpha)}(fg)$$

$$\leq M(a,b)\Gamma(1+\alpha)_0 I_1^{(\alpha)}(h_1h_2(1-\cdot)) + N(a,b)\Gamma(1+\alpha)_0 I_1^{(\alpha)}(h_1h_2)$$

and this is the desired result. ☑

***Remark* 3.8.** Clearly, if $h_1(t)=h_2(t)=t$ and $\alpha=1$ we obtain

$$2f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) - \frac{1}{2(b-a)}\int_a^b f(x)g(x)dx \leq \frac{1}{6}M(a,b)+\frac{1}{3}N(a,b)$$

which is the Theorem 1 given by Pachpatte in [24].

***Remark* 3.9.** For $s-$convex functions in second sense also we get a result showed by Kircmaci et al. in [15]. Making $h_1(t)=t$,$h_2(t)=t^s$ with $\alpha=1$ and $s\in(0,1]$ we obtain

$$2^{2s} f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) - \frac{1}{(b-a)}\int_a^b f(x)g(x)dx \leq \frac{M(a,b)}{(s+1)(s+2)}+\frac{N(a,b)}{s+2}.$$

4. Examples

Example 4.1. Let $a > 0, b > 0, x \in (0.\infty)$ and $a^{3\alpha}+b^{3\alpha} \leq 2^{\alpha}$. Then $a+b \leq 2$.

Proof. Let $f(x) = x^{3\alpha}$ for $x \in (0,\infty)$. It is easy to see that f is an *h-convex function* for $h(\lambda) = \lambda^{\alpha}$, for any $[a,b] \subset (0,\infty)$. Indeed

$$\begin{aligned}(ta+(1-t)b)^{3\alpha} &= t^{3\alpha}a^{3\alpha} + (1-t)^{3\alpha}\, b^{3\alpha} \\ &\leq t^{\alpha}a^{3\alpha} + (1-t)^{\alpha}\, b^{3\alpha}.\end{aligned}$$

Then

$$f\left(\frac{a+b}{2}\right) \leq h(1/2)f(a) + h(1/2)f(b)$$

in consequence

$$\begin{aligned}\left(\frac{a+b}{2}\right)^{3\alpha} &= \frac{a^{3\alpha}+b^{3\alpha}}{2^{3\alpha}} \leq \frac{a^{3\alpha}+b^{3\alpha}}{2^{\alpha}} \\ &= h(1/2)a^{3\alpha} + h(1/2)b^{3\alpha} \\ &= h(1/2)(a^{3\alpha}+b^{3\alpha}) \\ &\leq \frac{1}{2^{\alpha}}2^{\alpha} = 1^{\alpha}\end{aligned}$$

it follows that

$$\left(\frac{a+b}{2}\right)^{3} \leq 1$$

hence

$$\left(\frac{a+b}{2}\right) \leq 1$$

therefore, $a+b \leq 2$. ☑

Example 4.2. Let $0 < \alpha \leq 1$, $-1 < a,b < \infty$,with $a < b$, $t \in [0,1]$ and $f(x) = Ln(x+1)$ and $h(t) = (t+1)^{2\alpha}$ then f is *h-convex*. Indeed,

$$(Ln(a+1))^{\alpha} \leq Ln(ta+(1-t)b+1) \leq (Ln(b+1))^{\alpha}$$

it follows that

$$Ln(ta+(1-t)b+1) \leq (Ln(a+1))^{\alpha} + (Ln(b+1))^{\alpha}$$

and therefore

$$Ln(ta+(1-t)b+1) \leq (t+1)^{2\alpha}\,(Ln(a+1))^{\alpha} + (2-t)^{\alpha}\,(Ln(b+1))^{\alpha}$$

since for $t \in (0,1)$ we have $(t+1)^2 \geq 1$ y $(2-t)^2 \geq 1$.

Now, note that in the Hermite Hadamard inequality (3.3)

$$\begin{aligned}\frac{1}{\left(1-(-1)^{\alpha}\right)h(1/2)\Gamma\left(1+\alpha\right)}f(\frac{a+b}{2}) &\leq \frac{1}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{(\alpha)}f \\ &\leq \left(f\left(b\right)-(-1)^{\alpha}f\left(a\right)\right)\ {}_{0}I_{1}^{(\alpha)}h\end{aligned}$$

with $\alpha = 1$ we get

$$\begin{aligned}&\frac{1}{2\left(\frac{3}{2}\right)^{2}}Ln\left(\frac{a+b+1}{2}\right) \\ &\qquad\leq \frac{1}{(b-a)}\int_{a}^{b}Ln(x+1)dx \leq \left(Ln(a+1)+Ln(b+1)\right)\int_{0}^{1}(t+1)^{2}dt\end{aligned}$$

therefore we obtain the estimates

$$\frac{2}{9}Ln\left(\frac{a+b+1}{2}\right) \leq \frac{1}{(b-a)}\int_{a}^{b}Ln(x+1)dx \leq \frac{7}{3}Ln\left[(a+1)(b+1)\right].$$

References

[1] M. Klaričić Bakula and J. Pečarić, *Note of some hadamard type inequality*, Journal of Inequalities in Pure and Applied Mathematics **5** (2004), 119–124.

[2] M. Bombardelli and S. Varošanec, *Properties of h-convex functions related to the hermite-hadamard-fejér inequalities*, Computers and Mathematics with Applications **58** (2009), 1869–1877.

[3] W. W. Breckner, *Stetigkeitsaussagen für eineklasse verallgemeinerter konvexer funktionen in topologischen linearen räumen*, Pub. Inst. Math. **23** (1978), 13–20.

[4] S. S. Dragomir, *Inequalities of hermite-hadamard type for h-convex functions on linear spaces*, Proyecciones Journal of Mathematics **32** (2015), 323–341.

[5] S. S. Dragomir and S. Fitzpatrick, *The hadamard.s inequality for s-convex functions in the second sense*, Demonstratio Math. **32** (1999), 687–696.

[6] G. A. Edgar, *Measure, topology, and fractal geometry*, Springer-Verlag,New York, 1990.

[7] K. Falconer, *The geometry of fractal sets*, CambridgeUniversity Press,Cambridge, 1985.

[8] ______, *Fractal geometry*, John Wiley and Sons, Chichester, 1990.

[9] E. Gounova and V. Levin, *Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions. (russian) numerical mathematics and mathematical physics (russian)*, Numerical mathematics and mathematical physics (Russian). Moskov. Gos. Ped. Inst., Moscow **166** (1985), 138–142.

[10] ______, *Neravenstva dlja funkcii širokogo klassa, soderžaščego vypuklye, monotonnye,i nekotorye drugie vidy funkcii, in : Vyčislitel*, Mat. i. Mat. Fiz. Mežvuzov. Sb. Nauč. Trudov. MGPI. Moskva **32** (1985), 138–142.

[11] J. L. W Jensen, *Sur les fonctions convexes et le inequalitiés entre les valeurs moyennes*, Acta Math. **32** (1906), 175–193.

[12] Eder Kikianty, *Hermite-hadamard inequality in the geometry of banach spaces*, Tesis doctoral, School of Engineering and Science Faculty of Health, Engineering and Science Victoria University, 2010, The purpose of this thesis is to employ the Hermite-Hadamard inequality in studying the geometry of Banach spaces.

[13] A. Kiliçman and W. Saleh, *Notions of generalized s-convex functions on fractal sets*, Journal of Inequalities and Applications. Spingeropen Journal **312** (2015).

[14] ______, *Some generalizaed hermite-hadamard type integral inequalities for generalized s-convex functions on fractal sets*, Advances in Differences Equations. Spingeropen Journal **301** (2015).

[15] U. S. Kirmaci, M. K. Bakula, M. F. Özdemir, and J. Pečarić, *Hadamard type inequalities for s-convex functions*, Appl. Math. and Comp **193** (2007), 26–35.

[16] M. A Latif, *On some inequalities for h-convex functions*, Int. Journal of Math. Analysis **4** (2010), no. 30, 1473–1482.

[17] B. B. Mandelbrot, *The fractal geometry of nature*, Macmillan, New York, Ny, USA, 1983.

[18] N. Merentes and S. Rivas, *El desarrollo del concepto de función convexa*, XXVI Escuela venezolana de Matemáticas. Emalca - Venezuela, 2013.

[19] D. S. Mitrinović and I. B. Lačković, *Hermite and convexity*, Aequationes Math. **28** (1985), 225–232.

[20] D. S. Mitrinović and J. Pečarić, *Classical and new inequalities in analysis*, Kluwer Academic Publishers, Dordrecht /Boston /London, 1993.

[21] H. Mo and X. Sui, *Generalized s-convex functions on fractal sets*, ArXiv: 1405.0652.2v **28** (2014), 225–232.

[22] H. Mo, X. Sui, and D. Yu, *Generalized convex functions on fractal sets and two related inequalities*, Abstract and Applied Analisys **28** (2014), 225–232.

[23] M. Noor, *Hermite-hadamard integral inequalities fot log $\varphi-$ convex functions*, Nonlinear Anaisys Forum **13** (2008), no. 2, 119–124.

[24] B. G. Pachpatte, *On some inequalities for convex functions*, RGMIA. Res.Rep.Coll.6 **Coll. 6** (2003), 119–124.

[25] J. E. Pečarić, F. Proschan, and Y. L. Tong, *Convex functions, partial orderings, and statistical applications*, Academic Press, Inc., 1992.

[26] A. W. Roberts and D. Valberg, *Convex functions*, Pure and Applied Mathematics. Academic Press, 1973.

[27] S. Simič, *On a new converse of jensen's inequality*, Publications de L'Intitut Mathématique. Nouvelle serie **85** (2009), no. 99, 107–110.

[28] S. Varošanec, *On $h-$convexity*, J. Math. Anal. Appl. **326** (2007), 303–311.

[29] L. Wang, X. Ma, and L. Liu, *A note on some new refinements of jensen's inequality for convex functions*, Journal of Inequalities in Pure and Applied Mathematics **10** (2009), no. 2, 6 pp.

[30] X. J. Yang, *Advanced local fractional calculus an aplications*, World Science, NY, USA, 2012.

(Recibido en marzo de 2016. Aceptado en septiembre de 2016)

Departamento de Matemáticas
Facultad de Ciencias Naturales y Matemáticas
Escuela Superior politécnica del Litoral (ESPOL)
Campus, "Gustavo Galindo"
Km 30.5 Vía perimetral,
Guayaquil, Ecuador
e-mail: mjvivas@espol.edu.ec
Departamento de Matemáticas
Decanato de Ciencias y Tecnología
Universidad Centroccidental Lisandro Alvarado
Av. 20 , Av.Moran
Barquisimeto, Venezuela
e-mail: mvivas@ucla.edu.ve

Departamento de Técnicas Cuantitativas
Universidad Centroccidental Lisandro Alvarado
Decanato de Ciencias Económicas y Empresariales
Av. 20 , Av.Moran
Barquisimeto, Venezuela
e-mail: jorgehernandez@ucla.edu.ve

Escuela de Matemáticas
Universidad Central de Venezuela
Facultad de Ciencias
Av. Los Chaguaramos
Caracas, Venezuela
e-mail: nmerucv@gmail.com