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About and beyond the Lebesgue decomposition of a signed measure

Sobre el teorema de descomposición de Lebesgue de una medida con signo y un poco más

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ABSTRACT. We present a refinement of the Lebesgue decomposition of a signed measure. We also revisit the notion of mutual singularity for finite signed measures, explaining its connection with a notion of orthogonality introduced by R.C. JAMES.

Key words**:** Lebesgue Decomposition Theorem, Mutually Singular Signed Measures, Orthogonality.

RESUMEN. Presentamos un refinamiento de la descomposición de Lebesgue de una medida con signo. También revisitamos la noción de singularidad mutua para medidas con signo finitas, explicando su relación con una noción de ortogonalidad introducida por R. C. JAMES.

Palabras clave**:** Teorema de Descomposición de Lebesgue, Medidas con Signo Mutuamente Singulares, Ortogonalidad.

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1. Introduction

The main goal of this expository article is to present a refinement, not often found in the literature, of the Lebesgue decomposition of a signed measure. For the sake of completeness, we begin with a collection of preliminary definitions and results, including many comments, examples and counterexamples. We continue with a section dedicated specifically to the Lebesgue decomposition of a signed measure, where we also present a short account of its historical development. Next comes the centerpiece of our exposition, a refinement of the Lebesgue decomposition of a signed measure, which we prove in detail. Much can be said about the properties intervening in the formulation of this refinement. Although we will not say it all, we use the next section to follow up on some of it, including as well a brief historical commentary. In the last section, which can be viewed as an appendix, we revisit the notion of singularity for signed measures. Our purpose is twofold. First, we explain its connection with the notion of orthogonality in an inner product space. In this way, we justify the notation, $\nu_1 \perp \nu_2$, commonly used to indicate that two signed measures, ν_1 and ν_2, are mutually singular. Second, we show how a special class of singular signed measures, introduced in the previous section, serves to better illustrate what is in the complement of those finite signed measures having a density with respect to a fixed measure. The article ends with a list of references.

The inspiration for this article has been Professor TERENCE TAO'S *blog*, specifically his notes on measure theory [20].

2. Preliminary definitions and results

We begin by summarizing some of the definitions and results we need from measure theory. Other results will be stated at the appropriate time. Our purpose is to build a cohesive overview leading to the Lebesgue decomposition of a signed measure. As such, we will not emphasize proofs, but rather give precise references for them, specially for the proofs of the "big theorems". We will strive to state the results in their more general form relevant to our purpose. Unless otherwise noted, the material included in this section is mostly taken from [19].

Definition 1. A family Σ of subsets of a non-empty set X is called a σ-algebra if it satisfies the following three properties:

1. The empty set $\varnothing$ belongs to Σ.
2. If $E \in \Sigma$, then the complement $X \backslash E$ also belongs to Σ.
3. If $\{E_j\}_{j \geq 1} \subseteq \Sigma$, then $\bigcup_{j \geq 1} E_j \in \Sigma$.

The subsets of X that belong to Σ are called Σ-measurable.

With $\mathbb{R}^*$ we indicate the extended real number system consisting of the real numbers and the symbols $-\infty$ and $+\infty$, with the usual operations and order. We adopt the convention $0.(\pm\infty) = (\pm\infty).0 = 0$, but leave $(+\infty) + (-\infty)$ and $(-\infty) + (+\infty)$ undefined.

We recall that $\mathbb{R}^*$ becomes a compact metric space with the structure induced by the order preserving map $\varphi : \mathbb{R}^* \to [-1, 1]$, defined as

$$\begin{aligned} \varphi(-\infty) &= -1, \\ \varphi(+\infty) &= 1, \\ \varphi(x) &= \frac{x}{1+|x|} \text{ for } x \in \mathbb{R}. \end{aligned}$$

More on the algebraic and topological structures of $\mathbb{R}^*$ can be found in the very interesting article [25].

Given a σ-algebra Σ, we consider set functions $\nu : \Sigma \to \mathbb{R}^*$ that take at most one of the two values $-\infty$ and $+\infty$.

Definition 2. A set function $\nu : \Sigma \to \mathbb{R}^*$ is called a signed measure if

1. $\nu(\varnothing) = 0$ and
2. $\nu(\bigcup_{i\geq 1} E_i) = \sum_{i\geq 1} \nu(E_i)$ whenever $\{E_i\}_{i\geq 1} \subseteq \Sigma$ are pairwise disjoint.

As a consequence of 2) in Definition 2, the series $\sum_{i\geq 1} \nu(E_i)$ converges commutatively in $\mathbb{R}^*$ and, if $\nu(\bigcup_{i\geq 1} E_i)$ is finite, it converges absolutely in $\mathbb{R}$. If a set function ν satisfies 2) in Definition 2, we say that ν is countably additive.

Remark 1. Every countably additive set function is finitely additive as well. In fact, if $\{E_i\}_{1\leq i\leq n} \subseteq \Sigma$ are pairwise disjoint, it suffices to apply Definition 2 to the family $\{E_i\}_{i\geq 1}$ where $E_i = \varnothing$ for $i \geq n+1$.

We also observe that if E and $F \in \Sigma$, with $F \subseteq E$, and $\nu(E)$ is finite, meaning $\nu(E) \in \mathbb{R}$, then,

$$\nu(E - F) = \nu(E) - \nu(F).$$

Moreover, if $\nu(E)$ is finite for some $E \in \Sigma$, then $\nu(F)$ is finite for every $F \subseteq E$, $F \in \Sigma$. In fact, assuming that this statement is not true for some set F and writing

$$\nu(E) = \nu(F) + \nu(E - F),$$

we find a contradiction to the assumption that $\nu(E)$ is finite.

As a consequence, if $\nu(X)$ is finite, then $\nu : \Sigma \to \mathbb{R}$ and we say that the signed measure ν is finite.

If $\nu : \Sigma \to [0, +\infty]$, we say that ν is a *measure*, which will be finite if $\nu : \Sigma \to [0, +\infty)$.

The pair (X, Σ) is called a *measurable space*, while the triple (X, Σ, ν), where $\nu : \Sigma \to [0, +\infty]$ is a measure, is called a *measure space*.

Example 1. *1. The triple $(X, \mathcal{L}, \mu)$, where $\mathcal{L}$ is the Lebesgue σ-algebra on X and μ is the Lebesgue measure is called, naturally, the Lebesgue measure space, on X.*

2. *Given a measure space (X, Σ, μ) and given a μ-measurable function $f : X \to \mathbb{R}^*$, let f^+ and f^- be the positive part and the negative part of f, respectively. If at least one of $\int_E f^+ d\mu$ and $\int_E f^- d\mu$ is finite, we say ([19], p. 84), that f has a μ-integral on $E \in \Sigma$ and, then, we define*

$$\int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu.$$

If f has a μ-integral on E for every $E \in \Sigma$, then, the set function $\nu : \Sigma \to \mathbb{R}^*$, defined as

$$\nu (E) = \int_E f d\mu \tag{1}$$

is a signed measure, usually denoted $f d\mu$. To prove this claim, it suffices to show that the set function $\nu : \Sigma \to [0, +\infty]$, defined as $E \to \int_E f d\mu$, is a measure, when f is μ-measurable and non negative. Since it is obvious that $\nu (\varnothing) = 0$, we only need to prove that ν is countably additive, for which we approximate f with simple functions and use the Monotone Convergence Theorem ([19], p. 84).

Occasionally, it will be convenient to write (1) as $\nu (E) = \int_E f (x) \, dx$.

3. *If $\mu_1, \mu_2 : \Sigma \to [0, +\infty]$ are measures and one of them is finite, then $\mu_1 - \mu_2 : \Sigma \to \mathbb{R}^*$ is a signed measure.*

Definition 3. ([6], p. 123) Let $\nu : \Sigma \to \mathbb{R}^*$ be a signed measure. A set $A \subseteq X$ is called ν-positive if

1. $A \in \Sigma$ and
2. for every Σ-measurable set $B \subseteq A$, $0 \leq \nu (B) \leq +\infty$.

Definition 4. ([6], p. 123) Let $\nu : \Sigma \to \mathbb{R}^*$ be a signed measure. A set $A \subseteq X$ is called ν-negative if

1. $A \in \Sigma$ and
2. for every Σ-measurable set $B \subseteq A$, $-\infty \leq \nu (B) \leq 0$.

As a consequence of these two definitions, if a set A is both ν-positive and ν-negative, then for every Σ-measurable set $B \subseteq A$, $\nu (B) = 0$. If this is the case, we say that A is a ν-null set.

Definition 5. Given a signed measure $\nu : \Sigma \to \mathbb{R}^*$ and a Σ-measurable set A, the restriction of ν to A, denoted ν/A, is the set function, in fact a signed measure, defined on Σ as

$$(\nu/A) (E) = \nu (E \cap A) .$$

Alternatively, ν/A is the signed measure ν defined on the σ-algebra $\Sigma_A = \{E \cap A : E \in \Sigma\}$.

Let us observe that a Σ-measurable set A is ν-null exactly when $\nu/A = 0$. Furthermore, if ν is a measure, a Σ-measurable set A is ν-null when $\nu(A) = 0$. This is not the case for a signed measure, as shown by the following example:

Example 2. *Let $X = [-1, 1]$, let $\mathcal{L}$ be the Lebesgue σ-algebra on $[-1, 1]$ and let $\nu : \mathcal{L} \to \mathbb{R}$ be the signed measure defined as*

$$\nu(A) = \int_A x dx. \tag{2}$$

Then, $\nu\left([-\frac{1}{2}, \frac{1}{2}]\right) = 0$, but $\nu\left([0, \frac{1}{2}]\right) > 0$, showing that $[-\frac{1}{2}, \frac{1}{2}]$ is not ν-null.

Remark 2. Two signed measures, $\nu_1, \nu_2 : \Sigma \to \mathbb{R}^*$, are equal on a set $A \in \Sigma$ if $\nu_1(B) = \nu_2(B)$ for every Σ-measurable set $B \subseteq A$.

Theorem 1. *(Hahn decomposition), ([6], p. 124) Given a signed measure $\nu : \Sigma \to \mathbb{R}^*$, there is a partition $X = P \cup N$, where P and N are Σ-measurable, P is ν-positive and N is ν-negative.*

Any pair (P, N) of sets satisfying the conditions in Theorem 1 is called a Hahn decomposition of X relative to ν.

Remark 3. ([6], p. 124; [19], p. 33) Strictly speaking, the space X can have several Hahn decompositions relative to a given signed measure. For instance, if we take again $X = [-1, 1]$, if $\mathcal{L}$ is the Lebesgue σ-algebra on $[-1, 1]$ and $\nu : \Sigma \to \mathbb{R}$ is the signed measure defined as in (2), then $([0, 1], [-1, 0))$ and $((0, 1], [-1, 0])$ are both Hahn decompositions of $[-1, 1]$ relative to ν. Of course, the main point here is that $\{0\}$ is a ν-null set.

In general, if (P, N) is a Hahn decomposition of the space X relative to the signed measure ν and Z is a ν-null space, then $(P \cup Z, N \backslash Z)$ is another Hahn decomposition. However, we can say that Hahn decompositions are unique up to ν-null sets in the following sense:

If (P_1, N_1) and (P_2, N_2) are Hahn decompositions of X relative to ν, then for every Σ-measurable set E,

$$\nu(E \cap P_1) = \nu(E \cap P_2) \tag{3}$$

$$\nu(E \cap N_1) = \nu(E \cap N_2). \tag{4}$$

In 3) of Example 1, we constructed a signed measure as the difference of two measures, one of them being finite. The following result states that every signed measure can be written in this way.

Theorem 2. *(Jordan decomposition), ([6], p. 125) Every signed measure ν is equal to the difference of two unique measures, ν^+ and ν^-, at least one of which is finite.*

Remark 4. The construction of ν^+ and ν^- is a direct application of Theorem 1. Indeed, if (P, N) is any Hahn decomposition for X relative to ν, we define, for $E \in \Sigma$,

$$\nu^+(E) = \nu(E \cap P)$$
$$\nu^-(E) = -\nu(E \cap N).$$

The set functions ν^+ and ν^- are both signed measures, called, respectively, the positive and the negative variation of ν. Furthermore, since ν does not take both values, $+\infty$ and $-\infty$, either $\nu(P)$ is a, non negative, real number or $\nu(N)$ is a, non positive, real number. If $0 \leq \nu(P) < +\infty$, then $\nu^+ : \Sigma \to [0, +\infty)$. If $-\infty < \nu(N) \leq 0$, then $\nu^- : \Sigma \to [0, +\infty)$.

Moreover ([19], p. 29), if $E \in \Sigma$ is such that $\nu^+(E) = +\infty$, then $\nu(E) = +\infty$. Likewise, if $\nu^-(E) = +\infty$, then $\nu(E) = -\infty$.

In principle, since the proof of the Jordan decomposition is based on the Hahn decomposition, which, strictly speaking, is not unique, there is no reason to expect uniqueness in Theorem 2. To prove that ν^+ and ν^- are, indeed, unique, we introduce the notion of singularity of two measures. Later, this notion will be extended to signed measures.

Definition 6. Two measures, $\mu_1, \mu_2 : \Sigma \to [0, +\infty]$ are mutually singular, denoted $\mu_1 \perp \mu_2$, if there is a partition $X = A \cup B$ with $A, B \in \Sigma$, so that $\mu_1(B) = 0$ and $\mu_2(A) = 0$. In other words, if B is μ_1-null and A is μ_2-null.

According to Remark 4, the measures ν^+ and ν^- are mutually singular.

Now, let us consider two measures, λ_1 and λ_2, one of them finite, such that $\lambda_1 \perp \lambda_2$ and $\nu = \lambda_1 - \lambda_2$. Since $\lambda_1 \perp \lambda_2$, there is a partition $X = A \cup B$ with $A, B \in \Sigma$, so that $\lambda_1(B) = 0$ and $\lambda_2(A) = 0$. Moreover, the set A is ν-positive, while the set B is ν-negative. That is to say, (A, B) is a Hahn decomposition of X relative to ν. So, given $E \in \Sigma$,

$$\nu^+(E) = \nu(E \cap P) = \nu(E \cap A) = \lambda_1(E)$$
$$\nu^-(E) = -\nu(E \cap N) = -\nu(E \cap B) = \lambda_2(E)$$

where we have used (3) and (4).

Remark 5. In our presentation of the Hahn decomposition and the Jordan decomposition, we have followed [6]. It is possible to obtain, first, the Jordan decomposition of a signed measure and then, as a consequence, to prove the Hahn decomposition of the space relative to the signed measure. This is the approach taken in [19].

Proceeding with our overview, we now define the notion of total variation of a signed measure.

Definition 7. The total variation, denoted $|\nu|$, of a signed measure $\nu : \Sigma \to \mathbb{R}^*$, is the measure defined as

$$|\nu|(E) = \nu^+(E) + \nu^-(E),$$

for $E \in \Sigma$.

We claim that ([6], p. 126)

$$|\nu(E)| \le |\nu|(E), \tag{5}$$

for $E \in \Sigma$. To prove (5), let us observe that it is equivalent to

$$-|\nu|(E) \underset{\mathbf{(1)}}{\le} \nu(E) \underset{\mathbf{(2)}}{\le} |\nu|(E),$$

for $E \in \Sigma$. Since

$$\nu(E) = \nu^+(E) - \nu^-(E) \le \nu^+(E) + \nu^-(E) = |\nu|(E),$$

we have $\mathbf{(2)}$. To prove $\mathbf{(1)}$, we will show that $|\nu|(E) + \nu(E) \ge 0$.

If $\nu^-(E) \in \mathbb{R}$,

$$\begin{aligned}|\nu|(E) + \nu(E) &= \nu^+(E) + \nu^-(E) + \nu^+(E) - \nu^-(E)\\ &= 2\nu^+(E) \ge 0.\end{aligned}$$

If $\nu^-(E) = +\infty$, then $0 \le \nu^+(E) < +\infty$, so

$$\begin{aligned}\nu(E) &= \nu^+(E) - \nu^-(E) = -\nu^-(E)\\ &= -\nu^+(E) - \nu^-(E) = -|\nu|(E).\end{aligned}$$

Thus, we have $\mathbf{(1)}$.

Proposition 1. *([19], p. 30) The total variation $|\nu|$ can be defined as*

$$|\nu|(E) = \sup\left\{\sum_j |\nu(E_j)| : \{E_j\}_j \subseteq \Sigma, \textit{ any finite partition of } E\right\}. \tag{6}$$

From this proposition, it is possible to obtain another description of $|\nu|$ as the smallest of all the measures μ satisfying the condition $|\nu(E)| \le \mu(E)$, for $E \in \Sigma$. In fact, if μ satisfies this condition, then for each finite partition $\{E_j\}_j \subseteq \Sigma$ of E we can write

$$\sum_j |\nu(E_j)| \le \sum_j \mu(E_j) = \mu(E),$$

so, taking the supremum over all these partitions, $|\nu|(E) \le \mu(E)$.

From (6), it should be clear that given two signed measures, $\nu_1, \nu_2 : \Sigma \to \mathbb{R}^*$, if $\nu_1(E) + \nu_2(E)$ is well defined for all $E \in \Sigma$, then,

$$|\nu_1 + \nu_2| \le |\nu_1| + |\nu_2|.$$

Let us emphasize that the sum $\nu_1(E) + \nu_2(E)$ is well defined for $E \in \Sigma$ when we never encounter the combinations $(+\infty) + (-\infty)$ or $(-\infty) + (+\infty)$.

Example 3. *([19], p. 94) Following the notation in Example 1, if f has a μ-integral, then,*

$$\begin{aligned} \nu^{+} &= f^{+} d\mu \\ \nu^{-} &= f^{-} d\mu \\ |\nu| &= |f|\, d\mu. \end{aligned}$$

We are now ready to define the notion of absolute continuity of a signed measure with respect to another signed measure. We begin with the notion of absolute continuity of a measure with respect to another measure.

Definition 8. Given measures λ and μ, we say that λ is absolutely continuous with respect to μ, denoted $\lambda \ll \mu$, if $E \in \Sigma$ and $\mu(E) = 0$ imply $\lambda(E) = 0$.

Definition 9. Given signed measures λ and μ, we say that λ is absolutely continuous with respect to μ, denoted $\lambda \ll \mu$, if $|\lambda| \ll |\mu|$.

When μ is a measure, Definition 9 can be stated in several equivalent ways.

Proposition 2. *([4]. p. 138) If (X, Σ, μ) is a measure space and $\nu : \Sigma \to \mathbb{R}^*$ is a signed measure, the following statements are equivalent:*

1. $\nu \ll \mu$,
2. $\nu^{+} \ll \mu$ and $\nu^{-} \ll \mu$,
3. if $E \in \Sigma$ and $\mu(E) = 0$, then $\nu(F) = 0$ for all $F \subseteq E$, $F \in \Sigma$, and
4. if $E \in \Sigma$ and $\mu(E) = 0$, then $\nu(E) = 0$.

Proof. From Definitions 8 and 9, it should be clear that 1) is equivalent to 2) and that 2) implies 3). Moreover, 3) implies 4). So, it only remains to prove that 4) implies 1) or, equivalently, that 4) implies 2).

For this purpose, we consider a Hahn decomposition $X = P \cup N$, where P is ν-positive and N is ν-negative. If $E \in \Sigma$ and $\mu(E) = 0$, then $\mu(E \cap P) = 0$ and, according to 4), $\nu(E \cap P) = 0$. So,

$$\begin{aligned} 0 = \nu(E \cap P) &= \nu^{+}(E \cap P) - \nu^{-}(E \cap P) \\ &= \nu^{+}(E \cap P) = \nu^{+}(E). \end{aligned}$$

Similarly, since $\mu(E) = 0$ implies $\mu(E \cap N) = 0$ and, according to 4), $\nu(E \cap N) = 0$, we have

$$\begin{aligned} 0 = \nu(E \cap N) &= \nu^{+}(E \cap N) - \nu^{-}(E \cap N) \\ &= -\nu^{-}(E \cap N) = -\nu^{-}(E). \end{aligned}$$

So, $\nu^{+} \ll \mu$ and $\nu^{-} \ll \mu$.

This completes the proof. □

Example 4. *The signed measure*

$$\nu(E) = \int_E f d\mu \tag{7}$$

considered in 2) of Example 1, is absolutely continuous with respect to the measure μ. In fact, if $E \in \Sigma$ and $\mu(E) = 0$, then $\int_E f^+ d\mu = \int_E f^- d\mu = 0$. So, according to Example 3, $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

The next result proves that, under quite general assumptions, every signed measure ν, absolutely continuous with respect to a given measure μ, can be written as in (7). We begin with the following definition:

Definition 10. A measure $\mu : \Sigma \to [0, +\infty]$ is σ-finite if we can write $X = \bigcup_{j\geq 1} E_j$, with Σ-measurable sets E_j so that $\mu(E_j) < +\infty$ for all j.

It should be clear that every finite measure is σ-finite. The Lebesgue measure on $\mathbb{R}^n$ is an example of a σ-finite measure that is not finite. We can assume, and we will assume from now on, that the sets $\{E_j\}_{j\geq 1}$ are pairwise disjoint. If that is not the case, we only need to consider $F_1 = E_1$, $F_k = E_k \setminus \bigcup_{1\leq j\leq k-1} E_j$ for $k \geq 2$.

Theorem 3. *(Radon-Nikodym property), ([16], p. 238; [19], p. 133) Let (X, Σ, μ) be a measure space and assume that μ is σ-finite. Then, if $\nu : \Sigma \to \mathbb{R}^*$ is a signed measure absolutely continuous with respect to μ, there exists a μ-measurable function $f : X \to \mathbb{R}^*$ so that $\nu = f d\mu$. The function f is unique up to μ-null sets.*

Following the ideas of Calculus, ν can be seen as the indefinite integral of f with respect to ν. In this context, the function f is called the *Radon-Nikodym derivative* of ν with respect to μ, denoted $\frac{d\nu}{d\mu}$, or the density of ν in terms of μ.

Remark 6. ([19], p. 134) If the measure μ is not σ-finite, Theorem 3 might not be true, as shown by the following example:

Let us consider the Lebesgue measure space $([0,1], \mathcal{L}, \mu)$ and the counting measure $\nu : \mathcal{L} \to [0, +\infty]$ defined as

$$\nu(E) = \begin{cases} card(E) & \text{if } E \text{ is finite} \\ +\infty & \text{if } E \text{ is infinite.} \end{cases}$$

The Lebesgue measure μ is absolutely continuous with respect to the counting measure ν, since $\nu(E) = 0$ implies $E = \varnothing$ and thus, $\mu(E) = 0$. A function $f : [0,1] \to \mathbb{R}^*$ such that $\mu = f d\nu$ should be non-negative and ν-integrable on $[0,1]$. This implies that f must be finite. So, if E is the set defined as

$$E = \{x \in [0,1] : f(x) > 0\},$$

$\nu(E)$ is finite and

$$\mu([0,1]) = \int_E f d\nu = \mu(E) = 0,$$

which is a contradiction.

This completes the truly preliminary part in our overview of selected topics of measure theory. In the next section we go onto the central topic of this overview.

3. About the Lebesgue decomposition

We begin by extending Definition 6 and Definition 10 to signed measures.

Definition 11. Two signed measures $\nu_1, \nu_2 : \Sigma \to \mathbb{R}^*$ are mutually singular, denoted $\nu_1 \perp \nu_2$, if $|\nu_1|$ and $|\nu_2|$ are mutually singular.

Definition 12. A signed measure $\nu : \Sigma \to \mathbb{R}^*$ is σ-finite if the measure $|\nu|$ is σ-finite.

Remark 7. The relation of absolute continuity and the relation of mutual singularity are antithetical in the following sense: Given a measure space (X, Σ, μ) and given a signed measure $\nu : \Sigma \to \mathbb{R}^*$, if $\nu \perp \mu$ and $\nu \ll \mu$, then, ν must be identically zero. Indeed, given a partition $X = A \cup B$ with $A, B \in \Sigma$ so that $\mu(B) = 0$ and $\nu/A = 0$, according to Proposition 2, we must have $\nu/B = 0$ also.

In the following two lemmas we prove equivalent formulations of these definitions.

Lemma 1. *Given two signed measures $\nu_1, \nu_2 : \Sigma \to \mathbb{R}^*$, the following statements are equivalent:*

1. $\nu_1 \perp \nu_2$.
2. *there is a partition $X = A \cup B$ with $A, B \in \Sigma$ such that $\nu_1(B') = 0$ for all Σ-measurable $B' \subseteq B$, and $\nu_2(A') = 0$ for all Σ-measurable $A' \subseteq A$.*

Proof. If $|\nu_1| \perp |\nu_2|$, according to Definition 6, there is a partition $X = A \cup B$ with $A, B \in \Sigma$ so that $|\nu_1|(B) = 0$ and $|\nu_2|(A) = 0$. If $B' \subseteq B$ is Σ-measurable,

$$0 = |\nu_1|(B') = \nu_1^+(B') + \nu_1^-(B'),$$

so $\nu_1^+(B') = 0$ and $\nu_1^-(B') = 0$. Thus, $\nu_1(B') = 0$, with the same proof for ν_2 with A.

If we assume that 2) holds, using Proposition 1,

$$|\nu_1|(B) = \sup\left\{\sum_j |\nu(B_j)| : \{B_j\}_j \subseteq \Sigma, \text{any finite partition of } B\right\},$$

which implies $|\nu_1|(B) = 0$, since $B_j \subseteq B$ for all j. In the same way, using A, we show that $|\nu_2|(A) = 0$. This completes the proof. □

Lemma 2. *Given a signed measure $\nu : \Sigma \to \mathbb{R}^*$, the following statements are equivalent:*

1. *ν is σ-finite.*
2. *ν^+ and ν^- are σ-finite.*

3. *there is a partition* $X = \bigcup_{i\geq 1} E_i$ *where* E_i *is* Σ*-measurable and* $|\nu|(E_i) < +\infty$*, for all* $i \geq 1$.

Proof. That 1) implies 3) should be clear from Definition 12. To prove that 2) implies 1), let $X = \bigcup_{i\geq 1} E_i = \bigcup_{j\geq 1} F_j$, where E_i, F_j are Σ-measurable pairwise disjoint sets and $\nu^+(E_i) < +\infty$, $\nu^-(F_j) < +\infty$ for all $i, j \geq 1$. Then we can write $X = \bigcup_{i,j\geq 1} E_i \cap F_j$, where the sets $E_i \cap F_j$ are Σ-measurable and pairwise disjoint, and $|\nu|(E_i \cap F_j) < +\infty$ for all $i, j \geq 1$. Thus, the signed measure ν is σ-finite. To complete the proof, we now show that 3) implies 2). If we assume that 3) holds, according to the last part of Remark 4, $\nu^+(E_i) < +\infty$ and $\nu^-(E_i) < +\infty$. Thus, 2) holds. The proof is complete. □

Theorem 4. *(Lebesgue decomposition), ([19], p.141) Let* (X, Σ, μ) *be a measure space and let* $\nu : \Sigma \to \mathbb{R}^*$ *be a* σ*-finite signed measure. Then, there exist unique signed measures* $\nu_a, \nu_s : \Sigma \to \mathbb{R}^*$ *so that*

$$\begin{aligned} &\nu = \nu_a + \nu_s, \\ &\nu_a \ll \mu, \text{ and} \\ &\nu_s \perp \mu. \end{aligned}$$

Remark 8. ([19], p. 142) If the signed measure ν is not σ-finite, Theorem 4 will not be true in general. To prove this claim, we take the Lebesgue measure space $([0,1], \mathcal{L}, \mu)$ and the counting measure $\nu : \mathcal{L} \to [0, +\infty]$ defined as in Remark 6. It should be clear that ν is not σ-finite. We now show that ν cannot have a Lebesgue decomposition with respect to the Lebesgue measure μ. In fact, if such decomposition exists, there must be a partition $[0,1] = A \bigcup B$ so that $A, B \in \mathcal{L}$ and $\nu_s(B) = \mu(A) = 0$. For $x \in B$ fixed, we can write

$$1 = \nu(\{x\}) = \nu_a(\{x\}) + \nu_s(\{x\}) = 0,$$

because $\mu(\{x\}) = 0$ implies $\nu_a(\{x\}) = 0$ and, according to Lemma 1, $\nu_s(B) = 0$ implies $\nu_s(\{x\}) = 0$. Thus, we have arrived to a contradiction.

Remark 9. ([19], p. 142) The Jordan and Lebesgue decompositions are related in the following way: Given the Jordan decomposition $\nu = \nu^+ - \nu^-$ of a σ-finite signed measure $\nu : \Sigma \to \mathbb{R}^*$,

$$\begin{aligned} \nu_a &= (\nu^+)_a - (\nu^-)_a \\ \nu_s &= (\nu^+)_s - (\nu^-)_s . \end{aligned}$$

Besides the reference given in Theorem 4, there are a number of proofs of the Lebesgue decomposition theorem, specially for measures, as well as some interesting variations. For instance, see [3], [13], [14], [21], and [22].

We conclude this section with a brief account of the historical development of the Lebesgue decomposition theorem, which, as we will see, runs parallel to the development of the Radon-Nikodym theorem. Thus, it is with this theorem that we must begin. Its first version was formulated by LEBESGUE in 1910, using as reference the Lebesgue measure defined on the Borel sets of $\mathbb{R}$. In 1913, RADON, following the ideas of F. RIESZ and LEBESGUE, introduced the notion of measure as a "completely additive" function, that is a countably additive function, defined on the Lebesgue measurable sets of $\mathbb{R}^n$. RADON then proved that a measure can be written as the sum of a measure "with base μ", where μ is the Lebesgue measure, and a measure "foreign" to μ. In modern terminology, a measure with base μ means that it has a density with respect to μ, while a measure foreign to μ means that they are mutually singular. RADON also proved that a measure with base λ still has a density with respect to λ, where λ is a measure with base μ. Although RADON's construction of the density relied heavily on the topology of $\mathbb{R}^n$, it was consequently observed by FRÉCHET that most of RADON's results should remain true if the measure were to be defined on certain families of subsets of an arbitrary set, not necessarily the Lebesgue measurable sets of $\mathbb{R}^n$. Finally, it was NIKODYM who, in 1930, proved the existence of the density in the general case. Thus, the theorem should be known as Lebesgue-Radon-Nikodym. Let us add that NIKODYM's argument was greatly simplified by VON NEUMANN in 1940, using orthogonality properties of the space L^2 [23]. For more on these matters, as well as on the general historical development of measure theory, we refer to ([2], p. 227; [1], p. 105; [18]; [4], historical notes), and the references therein.

We now move on to the material that constitutes the core of our article.

4. Beyond the Lebesgue decomposition

As a preparation, we need to bring in a few ideas that follow up on Definition 5.

Definition 13. ([6], p. 140) A signed measure $\nu : \Sigma \to \mathbb{R}^*$ is concentrated on a Σ-measurable set A, if $\nu / (X \backslash A) = 0$, or equivalently, if $\nu / A = \nu$.

Equivalently, a signed measure is concentrated on A if $|\nu| (X \backslash A) = 0$.

We stress once again that $\nu (X \backslash A) = 0$ does not necessarily imply that ν is concentrated on A (Example 2).

If the signed measure ν is identically zero, we declare that it is trivially concentrated on the empty set.

Example 5. 1. *If $\nu : \Sigma \to \mathbb{R}^*$ is a signed measure with Jordan decomposition $\nu = \nu^+ - \nu^-$ and if $X = P \bigcup N$ is a Hahn decomposition of X relative to ν, then the measures $\nu^+$and ν^- are concentrated on P and N, respectively, according to Remark 4.*

2. *The Dirac measure $\delta_b : \mathcal{P}(\mathbb{R}^n) \to [0, +\infty)$ is defined as*

$$\delta_b (E) = \begin{cases} 1 & \text{if } b \in E \\ 0 & \text{if } b \notin E, \end{cases}$$

for all $E \in \mathcal{P}(\mathbb{R}^n)$, the σ-algebra of all the subsets of $\mathbb{R}^n$. Consequently, δ_b is concentrated on $\{b\} \subseteq \mathbb{R}^n$.

Now, given two sequences, $\{a_j\}_{j\geq 1}$ *and* $\{b_j\}_{j\geq 1}$*, where* $a_j \geq 0$ *for all* $j \geq 1$ *and* $\{b_j\}_{j\geq 1} \subseteq \mathbb{R}^n$*, the measure* $\nu : \mathcal{P}(\mathbb{R}^n) \to [0, +\infty]$ *defined as*

$$\nu = \sum_{j\geq 1} a_j \delta_{b_j}, \tag{8}$$

meaning

$$\nu(E) = \sum_{j\geq 1} a_j \delta_{b_j}(E),$$

is concentrated on the image of the sequence $\{b_j\}_{j\geq 1}$.

Definition 14. ([6], p. 12) A signed measure $\nu : \Sigma \to \mathbb{R}^*$ is discrete if the singletons belong to Σ and ν is concentrated on a countable set.

By countable, we mean a set that is finite or can be placed on a bijection with the set $\{1, 2, ...\}$ of natural numbers. A discrete signed measure is also called pure point [20].

Example 6. *The measures considered in 2) of Example 5 are discrete. As a matter of fact, if* (X, Σ, μ) *is a measure space and* $\nu : \Sigma \to \mathbb{R}^*$ *is a discrete signed measure concentrated on the set* $C = \{b_j\}_{j\geq 1} \in \Sigma$*, then* ν *is equal to the signed measure* δ_C *defined as* $\sum\limits_{j\geq 1} \nu(\{b_j\})\, \delta_{b_j}$*. Indeed, according to Remark 2, it suffices to observe that* $\nu/(X\backslash C) = \delta_C/(X\backslash C) = 0$ *and that* $\nu(\{b_j\}) = \delta_C(\{b_j\})$ *for all* $j \geq 1$.

When the underlying set X in Definition 14 has a topological structure, it is possible to talk about discrete sets. Let us recall that a set $E \subseteq X$ is discrete if for each $x \in E$ there is a neighborhood V_x so that $V_x \bigcap E = \{x\}$. The point of this digression is to remark that a discrete measure is not necessarily concentrated on a discrete set. For instance, if in 2) of Example 5 the sequence $\{b_j\}_{j\geq 1}$ is a bijection between the natural numbers and the vectors in $\mathbb{R}^n$ with rational coordinates, then the measure ν given by (8) is discrete but its support is not, since it is the set $\mathbb{Q}^n$.

Definition 15. A signed measure $\nu : \Sigma \to \mathbb{R}^*$ is continuous if the singletons belong to Σ and they are ν-null.

For instance, the Lebesgue measure $\mu : \mathcal{L} \to [0, +\infty]$ is continuous. Actually, there are other interesting examples of continuous signed measures, as well as several properties, which we think are worth looking into. We discuss some of them in the next section. For now, we will limit our exposition to definitions and results specifically needed for the refinement of the Lebesgue decomposition.

Remark 10. The concepts of discrete signed measure and continuous signed measure are antithetical in the following sense: If $\nu : \Sigma \to \mathbb{R}^*$ is a signed measure, both discrete and continuous, then ν is identically zero. Indeed, assume that ν is concentrated on the set $C = \{x_j\}_{j\geq 1}$. Since ν is continuous, $\nu(\{x_j\}) = 0$ for each $j \geq 1$. Thus, $\nu/C = 0$. Since $\nu/(X\backslash C) = 0$ by hypothesis, we conclude that ν is the zero measure.

The following result is a generalization of the classical Lebesgue decomposition theorem for the Lebesgue measure on the Borel sets of $\mathbb{R}$ (see [9], p. 182, and [10], p. 337).

Theorem 5. *(Beyond the Lebesgue decomposition), ([6], p. 140) Let (X, Σ) be a measurable space as in Remark 1, and assume that the singletons belong to Σ. Let $\mu : \Sigma \to [0, +\infty]$ be a continuous measure. Then, given a σ-finite signed measure $\nu : \Sigma \to \mathbb{R}^*$, there are unique σ-finite signed measures $\nu_1, \nu_2, \nu_3 : \Sigma \to \mathbb{R}^*$ so that*

1. $\nu = \nu_1 + \nu_2 + \nu_3$.
2. ν_1 *is discrete and* $\nu_1 \perp \mu$.
3. ν_2 *is continuous and* $\nu_2 \perp \mu$.
4. $\nu_3 \ll \mu$ *and, thus,* ν_3 *is continuous.*

Before proving this theorem, we need to establish two auxiliary results.

Lemma 3. *Let $\nu : \Sigma \to \mathbb{R}^*$ be a signed measure. Then ν is finite if and only if ν^+ and ν^- are finite.*

Proof. If ν^+ and ν^- are finite, it should be clear that ν must be finite, since $\nu = \nu^+ - \nu^-$. Conversely, as mentioned already in Remark 4, if $\nu^+(E) = +\infty$ for some Σ-measurable subset of X, then $\nu(E) = +\infty$. Likewise, if $\nu^-(E) = +\infty$, then $\nu(E) = -\infty$. This completes the proof. □

Lemma 4. *Given a signed measure $\nu : \Sigma \to \mathbb{R}^*$ and given $A \in \Sigma$,*

$$|\nu/A| = |\nu|/A.$$

Proof. For $E \in \Sigma$ we have, according to Proposition 1,

$$\begin{aligned}
|\nu/A|(E) &= \sup\left\{\sum_j |(\nu/A)(F_j)| : \{F_j\}_j \subseteq \Sigma, \text{finite partition of } E\right\} \\
&= \sup\left\{\sum_j \left|\nu\left(F_j \bigcap A\right)\right| : \{F_j\}_j \subseteq \Sigma, \text{finite partition of } E\right\} \\
&\le \sup\left\{\sum_j |\nu|\left(F_j \bigcap A\right) : \{F_j\}_j \subseteq \Sigma, \text{finite partition of } E\right\} \\
&\le |\nu|\left(E \bigcap A\right) = (|\nu|/A)(E).
\end{aligned}$$

Conversely, let us first assume that $(|\nu|/A)(E)$ is finite. Given $\varepsilon > 0$, there exists a finite partition $\{G_j\}_j$ of $E\bigcap A$ so that

$$\begin{aligned}(|\nu|/A)(E)-\varepsilon=|\nu|\left(E\bigcap A\right)-\varepsilon &\le \sum_j |\nu(G_j)| = \sum_j \left|\nu\left(G_j\bigcap A\right)\right| \\ &= \sum_j |(\nu/A)(G_j)| \\ &\le \sup\left\{\sum_j |(\nu/A)(F_j)| : \{F_j\}_j \subseteq \Sigma, \text{ finite partition of } E\right\} \\ &\le |\nu/A|(E).\end{aligned}$$

If $(|\nu|/A)(E) = +\infty$, then given $N \geq 1$, there exists a finite partition $\{G_j\}_j$ of $E\bigcap A$ so that

$$\begin{aligned}N &\le \sum_j |\nu(G_j)| = \sum_j \left|\nu\left(G_j\bigcap A\right)\right| \\ &= \sum_j |(\nu/A)(G_j)| \\ &\le \sup\left\{\sum_j |(\nu/A)(F_j)| : \{F_j\}_j \subseteq \Sigma, \text{ finite partition of } E\right\} \\ &\le |\nu/A|(E).\end{aligned}$$

In both cases, $(|\nu|/A)(E) \leq |\nu/A|(E)$. This completes the proof. □

Now we proceed with the proof of Theorem 5.

Proof. We start by considering the set $Y \subseteq X$ defined as

$$Y = \{x \in X : \nu(\{x\}) \neq 0\} = \{x \in X : |\nu|(\{x\}) \neq 0\}.$$

We claim that Y is countable. In fact, since ν is σ-finite, there is a partition $X = \bigcup_{j\geq 1} A_j$ with $A_j \in \Sigma$ and $|\nu(A_j)| < +\infty$. So, the restriction ν_j of ν to A_j, defined as $\nu_j(E) = \nu(E\bigcap A_j)$ for each $E \in \Sigma$, is a finite signed measure. We show that the set $Y\bigcap A_j$ is countable for each $j \geq 1$. To this purpose, we recall Lemma 4 and consider, for $n \geq 1$ fixed,

$$\left\{x \in A_j : |\nu|(\{x\}) \geq \frac{1}{n}\right\} = \left\{x \in A_j : |\nu_j|(\{x\}) \geq \frac{1}{n}\right\}.$$

This set has to be finite, otherwise, there would be an infinite subset $\{x_i\}_{i\geq 1}$ with $|\nu_j|(\{x_i\}) \geq \frac{1}{n}$ for all $i \geq 1$. Then, $|\nu_j|\left(\bigcup_{i\geq 1}\{x_i\}\right) = \sum_{i\geq 1} |\nu_j|(\{x_i\}) = +\infty$, which is a contradiction, according to Lemma 3.

Since

$$Y \bigcap A_j = \bigcup_{n \geq 1} \left\{ x \in A_j : |\nu_j| (\{x\}) \geq \frac{1}{n} \right\},$$

we can say that $Y \bigcap A_j$ is countable and thus

$$Y = \bigcup_{j \geq 1} \left(Y \bigcap A_j \right)$$

is countable as well.

We define ν_1 as the restriction of ν to the set Y. By construction, the signed measure ν_1 is discrete. Moreover, since μ is continuous, $\mu/Y = 0$. So, ν_1 and μ are mutually singular.

We consider next the signed measure that is the restriction of ν to the set $X \backslash Y$. Let us call it λ. Again by construction, λ is continuous. Using Remark 1 and Lemma 2, it should be clear that λ is σ-finite. Thus, we can invoke Theorem 4, the Lebesgue decomposition theorem, and write λ as $\nu_2 + \nu_3$, where $\nu_2 \perp \mu$ and $\nu_3 \ll \mu$. Since μ is continuous, ν_3 must be continuous also. We claim that ν_2 is continuous as well. To see it, we first observe that, by construction of ν_1, the signed measures ν and ν_1 have the same value on each singleton $\{x\}$. Indeed, $\nu_1 (\{x\})$ is zero exactly when $x \notin Y$ or $\nu (\{x\}) = 0$. So, given $x \in X$,

$$\nu_2 (\{x\}) = \nu (\{x\}) - \nu_1 (\{x\}) - \nu_3 (\{x\}) = 0.$$

Let us observe that $\nu (\{x\})$and $\nu_1 (\{x\})$ must be finite because ν is σ-finite.

To complete the proof, we need to show that the signed measures ν_1, ν_2 and ν_3 are unique. Suppose that we can write ν in two ways, $\nu_1 + \nu_2 + \nu_3$ and $\lambda_1 + \lambda_2 + \lambda_3$, with all the signed measures involved being σ-finite. We can assume that there is a partition $X = \bigcup_{j \geq 1} A_j$ with $A_j \in \Sigma$, so that $|\nu_i (A_j)| < +\infty$ and $|\lambda_i (A_j)| < +\infty$, for $i = 1, 2, 3$ and $j \geq 1$. Then, the equality $\nu_1 + \nu_2 - \lambda_1 - \lambda_2 = \lambda_3 - \nu_3$ on the σ-algebra $\Sigma_{A_j} = \{E \bigcap A_j : E \in \Sigma\}$, implies $\lambda_3 = \nu_3$ on Σ_{A_j}, according to Remark 7. Moreover, from $\nu_1 - \lambda_1 = \lambda_2 - \nu_2$, Remark 10 implies that $\nu_1 = \lambda_1$ and $\nu_2 = \lambda_2$, on Σ_{A_j}. Finally, since $\{A_j\}_{j \geq 1}$ is a partition of X, we can conclude that $\nu_1 = \lambda_1$ and $\lambda_2 = \nu_2$.

Thus, the proof is complete. □

5. On continuous measures and the like

As mentioned before, this section will be dedicated to look deeper into the notion of continuous measure. Since much can be said about this subject, we will limit ourselves to review a few properties and related concepts. To begin, we consider a simple property of the Lebesgue measure, that hints at the meaning of the word continuous, in the context of measure theory.

Proposition 3. *Consider the Lebesgue measure space* $(\mathbb{R}^n, \mathcal{L}, \mu)$. *Given* $E \in \mathcal{L}$ *with* $\mu (E) > 0$, *there exists an* $\mathcal{L}$*-measurable set* $F \subseteq E$ *so that* $0 < \mu (F) < \mu (E)$.

Proof. Given $r > 0$ to be chosen, consider the covering of $\mathbb{R}^n$ by open balls $B_{x,r}$ of center x and radius r, for $x \in \mathbb{R}^n$, and consider a countable subcovering, $\{B_{x_j,r}\}_{j\geq 1}$. Then, $\{B_{x_j,r} \bigcap E\}_{j\geq 1}$ is a countable covering of E, so

$$0 < \mu(E) \leq \sum_{j\geq 1} \mu\left(B_{x_j,r} \bigcap E\right).$$

Since $\mu(E) > 0$, there is $j_0 \geq 1$ so that $\mu\left(B_{x_{j_0},r} \bigcap E\right) > 0$. Moreover, $\mu\left(B_{x_{j_0},r} \bigcap E\right) \leq \mu\left(B_{x_{j_0},r}\right) = c_n r^n$ for some $c_n > 0$. If we choose r so that $0 < r < \left(\frac{\mu(E)}{c_n}\right)^{1/n}$, we will have

$$0 < \mu\left(B_{x_{j_0},r} \bigcap E\right) < \mu(E).$$

This completes the proof. □

Definition 16. ([7], p. 645) Given a measure $\nu : \Sigma \to [0, +\infty]$ and given $A \in \Sigma$, we say that A is a ν-atom if $\nu(A) > 0$ and for every Σ-measurable set $B \subseteq A$, is $\nu(B) = 0$ or $\nu(B) = \nu(A)$. A measure without atoms is called atomless.

Every non identically zero discrete measure ν has to have atoms. In fact if $\nu(\{x\}) \neq 0$, then $\{x\}$ is a ν-atom.

Some sources refer to a measure without atoms as non atomic or not atomic. However, the words “non atomic” can have other meanings in measure theory ([6], p. 290), while the words “not atomic” have a very precise meaning in computer programming [24], so we prefer to use the word atomless.

Definition 16 can be extended to signed measures in various ways, for instance, by saying that $A \in \Sigma$ is a ν-atom if $\nu(A) \neq 0$ and for each $B \subseteq A$, $B \in \Sigma$, $\nu(B) = 0$ or $|\nu(B)| = |\nu(A)|$ ([8], p. 20). In what follows, we will limit ourselves to the consideration of measures.

Example 7. *Given the Lebesgue measure space* $(\mathbb{R}^n, \mathcal{L}, \mu)$*, the measure* μ *is atomless, according to Proposition 3.*

Remark 11. Let (X, Σ) be a measurable space and assume that the singletons belong to Σ. Then every atomless measure $\nu : \Sigma \to [0, +\infty]$ is continuous. In fact, every singleton $\{x\}$ for which $\nu(\{x\}) > 0$ must be a ν-atom. The converse is not true in general, as shown by the following example:

Let X be an uncountable set and let

$$\Sigma = \{E \subseteq X : E \text{ is countable or } X \backslash E \text{ is countable}\}.$$

The family Σ is a σ-algebra. On Σ define the set function, in fact the measure,

$$\mu(E) = \begin{cases} 0 & \text{if } E \text{ is countable} \\ 1 & \text{if } X \backslash E \text{ is countable.} \end{cases}$$

The singletons belong to Σ and are μ-null sets. However, μ is not atomless, the sets E for which $X \backslash E$ is countable being the μ-atoms.

Lemma 5. *If $\nu : \Sigma \to [0, +\infty]$ is σ-finite and $A \in \Sigma$ is a ν-atom, then $\nu(A) < +\infty$.*

Proof. We can write $A = \bigcup_{j \geq 1} A_j$, disjoint union of Σ-measurable sets with $\nu(A_j) < +\infty$. For each $j \geq 1$, $\nu(A_j) = 0$ or $\nu(A_j) = \nu(A)$. Since $\nu(A) = \sum_{j \geq 1} \nu(A_j)$ and $\nu(A) > 0$, there exists $j_0 \geq 1$ so that $\nu(A_{j_0}) > 0$. Thus, $\nu(A) = \nu(A_{j_0}) < +\infty$. The proof is complete. □

The converse of this lemma is not true, in general. For instance, consider the counting measure as in Remark 6, but defined on the subsets of an uncountable set. For this measure, the atoms are exactly all the singletons, with measure one.

Proposition 3 has a very interesting extension for finite atomless measures.

Proposition 4. *([7], p. 645) Let (X, Σ) be a measurable space. If $\nu : \Sigma \to [0, +\infty)$ is a non identically zero atomless finite measure, for each real number c, $0 < c < \nu(X)$, there exists $E \in \Sigma$ so that $\nu(X) = c$.*

This proposition can be viewed as an intermediate value property. For the somewhat lengthy proof, we refer to ([7], p. 645). The interesting point is that, for the subclass of atomless measures, the word continuous should be interpreted as meaning that the measure takes a continuum of values.

The first version of such a result was proved by SIERPIŃSKI in an article published in 1922 [17]. In this article, he fixes a bounded subset E_0 of $\mathbb{R}^n$ and considers the class $\mathcal{L}(E_0)$ of those Lebesgue measurable sets $E \subseteq E_0$. He works with a function, that he denotes f, defined on $\mathcal{L}(E_0)$ with real values. By "*fonction d'ensemble additive et continue*", SIERPIŃSKI means that f is finitely additive, and continuous in the sense that there is $\lim f(E) = 0$ as the diameter of the set E goes to zero. Then, SIERPIŃSKI proceeds to show that f satisfies an intermediate value property that he states in the following way: Given sets $E_1, E_2 \in \mathcal{L}(F_0)$ and given a real number t, $0 \leq t \leq 1$, there exists a set $E = E(t) \in \mathcal{L}(E_0)$ such that

$$f(E) = (1 - t) f(E_1) + t f(E_2).$$

SIERPIŃSKI goes on to observing that such function f is zero on the singletons, that is to say, that it is continuous in the sense of Definition 15.

Remark 12. Proposition 4 can be easily extended to non identically zero atomless σ-finite measures. In fact, let $X = \bigcup_{j \geq 1} A_j$, a disjoint union of Σ-measurable sets with $\nu(A_j) < +\infty$. We can assume that $\nu(X) = +\infty$. Then, given any real number $r > 0$, there must exist $N \geq 1$ so that $r < \sum_{1 \leq j \leq N} \nu(A_j)$. Now, we consider the restriction of ν to $\Sigma_{\bigcup_{1 \leq j \leq N} A_j}$ as in Definition 5. This measure is finite, so there exists $E \in \Sigma$ such that

$$r = \nu\left[E \bigcap \left(\bigcup_{j=1}^{N} A_j\right)\right].$$

Proposition 4 is not true in general, if we allow the measure to have atoms. For instance, consider the counting measure, defined on the subsets of a non empty finite set.

We conclude this section with the promised example of an interesting class of continuous signed measures.

Example 8. *Let us consider, for* $n \geq 2$*, the measure spaces* $(\mathbb{R}^n, \mathcal{B}_n, \mu_n)$ *and* $\left(\mathbb{R}^{n-1}, \mathcal{B}_{n-1}, \mu_{n-1}\right)$*, where* $\mathcal{B}_n$ *and* $\mathcal{B}_{n-1}$ *are the Borel* σ*-algebras,* μ_n *and* μ_{n-1} *are the Lebesgue measures.*

Given a $\mathcal{B}_{n-1}$*-measurable function* $f : \mathbb{R}^{n-1} \to \mathbb{R}^*$ *having an integral with respect to* μ_{n-1} *((2) in Example 1), we define a set function* $\lambda : \mathcal{B}_n \to \mathbb{R}^*$ *as*

$$\lambda(E) = \int_{\mathbb{R}^{n-1}} f(x') \chi_E(x', 0)\, d\mu_{n-1},$$

where χ_E *denotes the characteristic function of* E*. This set function* λ *is a signed measure and, furthermore, it is continuous. Moreover,* λ *and* μ_n *are mutually singular. In fact, if* $\mathbb{X}_{n-1} = \left\{(x', 0) : x' \in \mathbb{R}^{n-1}\right\}$*,*

$$\mu_n(\mathbb{X}_{n-1}) = 0 \text{ and } \lambda(\mathbb{R}^n \backslash \mathbb{X}_{n-1}) = 0.$$

The same idea works with any subspace $\mathbb{X}_k$ *of* $\mathbb{R}^n$ *for* $0 < k < n$*. That is to say, we have a class of signed measures that are extensions of lower dimensional Lebesgue measures and are continuous, and mutually singular with* μ_n*.*

In the next, and last, section, we will revisit this example, to illustrate its interest.

6. Where we show that mutually singular finite signed measures are sort of orthogonal

We fix again a measurable space (X, Σ). In what follows we will consider the space, denoted $\mathcal{M}$, of finite signed measures $\nu : \Sigma \to \mathbb{R}$.

Proposition 5. *The space* $\mathcal{M}$ *becomes a real Banach space if we define*

$$\|\nu\|_{\mathcal{M}} = |\nu|(X).$$

Proof. From the definition and the properties of the total variation $|\nu|$, it should be clear that $\|\cdot\|_{\mathcal{M}}$ is a norm. As for the completeness, we will give a fairly direct proof of this known result.

Let $\{\nu_j\}_{j\geq 1}$ be a Cauchy sequence in $\mathcal{M}$. That is to say, given $\varepsilon > 0$, there is $j_0 = j_0(\varepsilon) \geq 1$ so

$$\|\nu_j - \nu_l\|_{\mathcal{M}} < \varepsilon,$$

for $j, l \geq j_0$. This implies that

$$|\nu_j(E) - \nu_l(E)| = |(\nu_j - \nu_l)(E)| \leq |\nu_j - \nu_l|(X) < \varepsilon,$$

for every $E \in \Sigma$, so the sequence of set functions $\nu_j : \Sigma \to \mathbb{R}$ is Cauchy, uniformly on Σ. Moreover, for each $E \in \Sigma$, the real sequence $\{\nu_j(E)\}_{j\geq 1}$ is a Cauchy sequence, so it has limit. We define

$$\nu(E) = \lim_{j\to\infty} \nu_j(E). \tag{9}$$

Since the sequence $\{\nu_j\}_{j\geq 1}$ is Cauchy, uniformly on Σ, the limit in (9) is uniform on Σ. We claim that the set function $\nu : \Sigma \to \mathbb{R}$ belongs to $\mathcal{M}$. Firstly, it is clear that $\nu(\emptyset) = 0$. To prove that ν is countably additive, we begin by observing that ν is finitely additive, from its definition. In fact, if $\{E_k\}_{1\leq k\leq p}$ is any finite family of pairwise disjoint sets in Σ,

$$\begin{aligned}\nu\left(\bigcup_{k=1}^{p} E_k\right) &= \lim_{j\to\infty} \nu_j\left(\bigcup_{k=1}^{p} E_k\right) = \lim_{j\to\infty} \sum_{k=1}^{p} \nu_j(E_k)\\ &= \sum_{k=1}^{p} \lim_{j\to\infty} \nu_j(E_k) = \sum_{k=1}^{p} \nu(E_k).\end{aligned}$$

Let us consider next a family of pairwise disjoint sets in Σ, $\{E_k\}_{k\geq 1}$, and let us write $E = \bigcup_{k\geq 1} E_k$. For $p \geq 1$ and $j \geq 1$ to be chosen later, we have

$$\begin{aligned}\left|\nu(E) - \sum_{k=1}^{p} \nu(E_k)\right| &\leq |\nu(E) - \nu_j(E)| + \left|\sum_{k=1}^{p} (\nu_j(E_k) - \nu(E_k))\right|\\ &+ \left|\sum_{k\geq p+1} \nu_j(E_k)\right| = (1) + (2) + (3).\end{aligned}$$

Let us estimate each of these three terms. For (1), $|\nu(E) - \nu_j(E)| < \varepsilon/3$, for $j \geq j_0 = j_0(\varepsilon)$, independently of E. We then fix $j = j_0$ in the other two terms. For (3), since $\sum_{k\geq 1} \nu_{j_0}(E_k)$ converges, to $\nu_{j_0}(E)$, $\left|\sum_{k\geq p+1} \nu_{j_0}(E_k)\right| < \varepsilon/3$, for $p \geq p_0$. Finally, for $p \geq p_0$, we can write (2) as

$$\left|\sum_{k=1}^{p} (\nu_{j_0}(E_k) - \nu(E_k))\right| = \left|(\nu_{j_0} - \nu)\left(\bigcup_{k=1}^{p} E_k\right)\right| < \varepsilon/3,$$

since we already observed that the convergence is uniform on Σ. Thus, $\nu \in \mathcal{M}$.

The last step is to prove that $\{\nu_j\}_{j\geq 1}$ converges to ν in $\mathcal{M}$, for which we use the right hand side of the following inequality (see [19], p. 30):

$$\sup_{E\in\Sigma} |\nu_j(E) - \nu(E)| \leq |\nu_j - \nu|(X) \leq 2 \sup_{E\in\Sigma} |\nu_j(E) - \nu(E)|.$$

Given $\varepsilon > 0$, if we use again the uniform convergence of $\{\nu_j\}_{j\geq 1}$ to ν, we have $|\nu_j(E) - \nu(E)| < \varepsilon/2$, for $j \geq j_0(\varepsilon)$ and for all $E \in \Sigma$. So, the proof is complete. □

After this preparatory work, we are ready to tackle the orthogonality issue. We begin with a definition.

Definition 17. ([11], p. 292) Given a real normed space $(N, \|\cdot\|_N)$, and given $u, v \in N$, we say that u is orthogonal to v, denoted $u \perp v$, if

$$\|u+v\|_N = \|u-v\|_N . \tag{10}$$

Notice that this definition gives a symmetric relation in u and v, so we can say that u and v are orthogonal. The following result justifies the use of the word "orthogonal" in Definition 17.

Proposition 6. *([11], p. 292) Let $(H, \langle , \rangle_H)$ be a real inner product space. Then, given $u, v \in H$, $v \neq 0$, the following statements are equivalent:*

1. $\|u+v\|_H = \|u-v\|_H$.
2. $\langle u, v\rangle_H = 0$.

Proof. We start by writing out the expression $\|u+v\|_H^2 - \|u-v\|_H^2$.

$$\begin{aligned}\|u+v\|_H^2 - \|u-v\|_H^2 &= \langle u+v, u+v\rangle_H - \langle u-v, u-v\rangle_H \\ &= \langle u, u\rangle_H + 2\langle u, v\rangle_H + \langle v, v\rangle_H - \langle u, u\rangle_H \\ &+ 2\langle u, v\rangle_H - \langle v, v\rangle_H \\ &= 4\langle u, v\rangle_H .\end{aligned}$$

Thus, if 2) holds, then $\|u+v\|_H = \|u-v\|_H$. Conversely, if 1) holds, then $\langle u, v\rangle_H = 0$.

This completes the proof. □

Proposition 7. *([15], p. 165) If $\nu_1, \nu_2 \in \mathcal{M}$ and $\nu_1 \perp \nu_2$ in the sense that ν_1 and ν_2 are mutually singular, then $\nu_1 \perp \nu_2$ in the sense of Definition 17. However, the converse is not true.*

Proof. According to ([15], p. 165), ν_1 and ν_2 are mutually singular if and only if $|\nu_1+\nu_2| = |\nu_1-\nu_2|$. This clearly implies that $\|\nu_1+\nu_2\|_{\mathcal{M}} = \|\nu_1-\nu_2\|_{\mathcal{M}}$. Thus, ν_1 and ν_2 are orthogonal in the sense of Definition 17.

To see that the converse is not true, we consider the following example:

Let $f(x) = \sin x$ and $g(x) = \cos x$ defined on $[0, \pi]$. On the measure space $([0, \pi], \mathcal{L})$, define the signed measures $v_1 = f dx$ and $v_2 = g dx$.

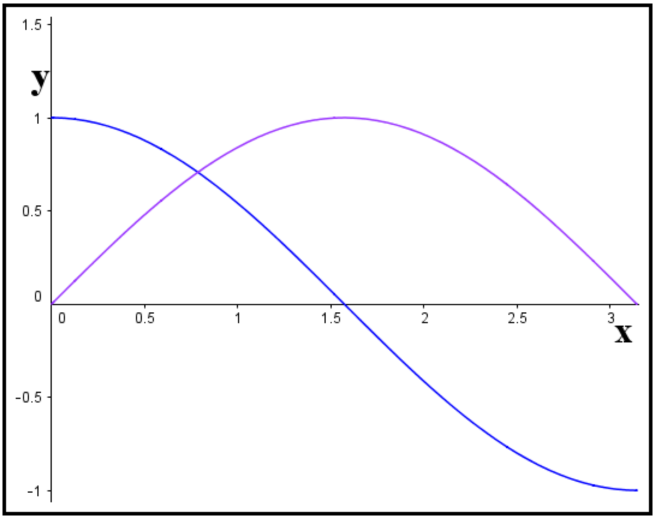

Figure 1. $\sin x$ and $\cos x$.

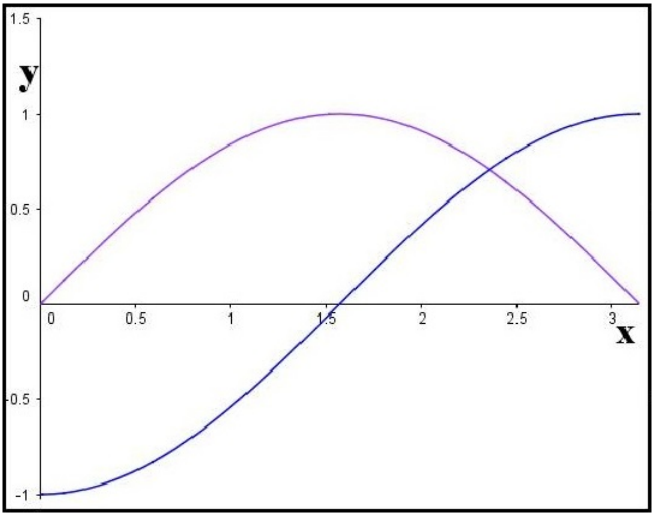

Figure 2. $\sin x$ and $-\cos x$.

As these two graphics show,

$$\int_0^\pi |f+g|(x)\,dx = \int_0^\pi |f-g|(x)\,dx,$$

or

$$\|\upsilon_1+\upsilon_2\|_{\mathcal{M}} = \|\upsilon_1-\upsilon_2\|_{\mathcal{M}}.$$

However, for $E=\left[0,\frac{\pi}{4}\right]$ we have

$$|\upsilon_1+\upsilon_2|(E) = \int_0^{\pi/4} |f+g|(x)\,dx = \int_0^{\pi/4} \sin x dx + \int_0^{\pi/4} \cos x dx = 1,$$

while

$$|\mu - \upsilon|(E) = \int_0^{\pi/4} |f - g|(x)\, dx < 1.$$

This completes the proof. □

To conclude our glance at the orthogonality of mutually singular finite signed measures, let us say that there is ample evidence of this connection, starting with VON NEUMANN's proof of the Lebesgue decomposition theorem [23]. Other references include [21]. Moreover, combining the Lebesgue decomposition theorem and the Radon-Nikodym theorem for a finite signed measure ν and a σ-finite measure μ, we obtain a very simple version of the projection theorem [5].

There are other formulations of orthogonality, all equivalent in the context of real inner product spaces (see, for example [11], [12]). However, the most interesting formulations are those preserving the property that in every two-dimensional subspace there exist non zero orthogonal vectors [11]. The notion of orthogonality given in Definition 17 satisfies such property [11] .

We close our exposition with another look at the class of continuous signed measures described in Example 8. To this purpose, we consider, for $n \geq 2$, the Borel measure space $(\mathbb{R}^n, \mathcal{B}_n, \mu_n)$, and the Banach space $\mathcal{M}$ of finite signed measures $\nu : \mathcal{B}_n \to \mathbb{R}$. We define the subspace $\mathcal{N}$ of $\mathcal{M}$ as

$$\mathcal{N} = \{\nu \in \mathcal{M} : \nu \ll \mu_n\}.$$

According to Theorem 3, the Radon-Nikodym property, each signed measure in $\mathcal{N}$ has the form $f d\mu_n$, for $f \in L^1(\mu_n)$ uniquely defined up to a μ_n-null set. Furthermore, according to Example 3,

$$\|f d\mu_n\|_{\mathcal{M}} = \int_{\mathbb{R}^n} |f|\, d\mu_n = \|f\|_{L^1(\mu_n)}.$$

Thus, $\mathcal{N}$ is isometrically isomorphic to the Banach space $L^1(\mu_n)$ and, as a consequence, it is a closed subspace of $\mathcal{M}$. As for examples of signed measures in the complement $\mathcal{M}\backslash\mathcal{N}$, quite often these examples are discrete measures such as those presented in Example 5 and Example 6. However, the class discussed in Example 8, provides an entirely different type of signed measure belonging to $\mathcal{M}\backslash\mathcal{N}$. This is a feature that makes the class quite interesting.

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