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Multiplication operators in variable Lebesgue spaces

Operador multiplicación en los espacios de Lebesgue con exponente variable

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Abstract. In this note we will characterize the boundedness, invertibility, compactness and closedness of the range of multiplication operators on variable Lebesgue spaces.

Key words and phrases. Multiplication operator, variable Lebesgue spaces, compactness.

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Resumen. En esta nota vamos a caracterizar los operadores multiplicación que son continuos, invertibles y que tienen rango cerrados sobre los espacios de Lebesgue con exponente variable.

Palabras y frases clave. Operador multiplicación, espacios de Lebesgue variables, compacidad.

1. Introduction

Variable Lebesgue spaces are a generalization of Lebesgue spaces where we allow the exponent to be a measurable function and thus the exponent may vary. It seems that the first occurence in the literature is in the 1931 paper of Orlicz [29]. The seminal work on this field is the 1991 paper of Kováčik and Rákosník [27] where many basic properties of Lebesgue and Sobolev spaces were shown. To see a more detailed history of such spaces see, e.g., [17, §1.1]. These variable exponent function spaces have a wide variety of applications, e.g., in the modeling of electrorheological fluids [4, 5, 31] as well as thermorheological fluids [6], in the study of image processing [1, 2, 10, 11, 14, 15, 34] and in differential equations with non-standard growth [23, 28]. For details on variable Lebesgue spaces one can refer to [16, 17, 21, 27, 30] and the references therein.

The multiplication operator, defined roughly speaking as the pointwise multiplication by a real-valued measurable function, is a well-studied transformation. This operator received considerable attention over the past several decades, specially on Lebesgue and Bergman spaces and they played an important role in the study of operators on Hilbert spaces. For more details on these operators we refer to [3, 9, 18, 22, 32]. Studies of the multiplication operator on various spaces can be seen, e.g. [7, 8, 13, 12, 25, 26, 33], in particular on L^p space in [25, 33], on Orlicz space in [26], on Lorentz space in [7] and on Lorentz-Bochner space in [8, 20]. It is natural to extend the study to variable Lebesgue spaces.

The main goal of the present note is to establish boundedness, invertibility, compactness and closedness of multiplication operators in the framework of variable Lebesgue spaces $L^{p(\cdot)}(X, \mu)$.

2. Preliminaries

2.1. On Lebesgue spaces with variable exponent

The basics on variable Lebesgue spaces may be found in the monographs [16, 17] (see also [24, 27]), but we recall here some necessary definitions. Let (X, Σ, μ) be a σ-finite, complete measure space. For $A \subset X$ we put $p_A^+ := \operatorname{ess\,sup}_{x\in A} p(x)$ and $p_A^- := \operatorname{ess\,inf}_{x\in A} p(x)$; we use the abbreviations $p^+ = p_X^+$ and $p^- = p_X^-$. For a measurable function $p : X \to [1, \infty)$, we call it a *variable exponent*, and define the set of all variable exponents with $p^+ < \infty$ as $\mathcal{P}(X, \mu)$. In this note, all the variable exponents are tacitly assumed to belong to the class $\mathcal{P}(X, \mu)$.

For a real-valued μ-measurable function $\varphi : X \to \mathbb{R}$ we define the *modular*

$$\rho_{p(\cdot)}(\varphi) := \int_X |\varphi(x)|^{p(x)} \, \mathrm{d}\mu(x)$$

and the *Luxemburg norm*

$$\|\varphi\|_{L^{p(\cdot)}(X,\mu)} := \inf\left\{\lambda > 0 : \rho_{p(\cdot)}\left(\frac{\varphi}{\lambda}\right) \leqslant 1\right\}. \tag{1}$$

Definition 2.1. Let $p \in \mathcal{P}(X, \mu)$. The *variable Lebesgue space* $L^{p(\cdot)}(X, \mu)$ is introduced as the set of all real-valued μ-measurable functions $\varphi : X \to \mathbb{R}$ for which $\rho_{p(\cdot)}(\varphi) < \infty$. Equipped with the Luxemburg norm (1) this is a Banach space.

We gather here some useful properties of variable exponent Lebesgue spaces, see [17, p. 77]

***Remark* 2.2.** We say that a function is *simple* if it is the linear combination of indicator functions of measurable sets with finite measure, $\sum_{i=1}^{k} s_i \chi_{A_i}(x)$ with $\mu(A_1) < \infty, \dots, \mu(A_k) < \infty$ and $s_1, \dots, s_k \in \mathbb{R}$. We denote the set of simple functions by $S(X, \mu)$.

Proposition 2.3. *Let $p \in \mathcal{P}(X, \mu)$. Then the set of simple functions $S(X, \mu)$ is contained in $L^{p(\cdot)}(X, \mu)$ and*

$$\min\{1, \mu(E)\} \leqslant \|\chi_E\|_{L^{p(\cdot)}(X,\mu)} \leqslant \max\{1, \mu(E)\}$$

for every measurable set $E \subset X$.

Proposition 2.4. *If $p \in \mathcal{P}(X, \mu)$, then the set of simple functions is dense in $L^{p(\cdot)}(X, \mu)$.*

***Remark* 2.5.** The previous proposition is explicitly stated in [17, Corollary 3.4.10] for a domain $\Omega \subset \mathbb{R}^n$, but the result can be stated in full generality as done above, since the result is a corollary of Theorem 2.5.9 and Theorem 3.4.1(c) from [17] which are given for a σ-finite, complete measure spaces (X, Σ, μ).

Proposition 2.6. *The variable Lebesgue space $L^{p(\cdot)}(X, \mu)$ is circular, solid, satisfies Fatou's lemma (for the norm) and has the Fatou property, namely:*

circular $\|f\|_{L^{p(\cdot)}(X,\mu)} = \||f|\|_{L^{p(\cdot)}(X,\mu)}$ *for all $f \in L^{p(\cdot)}(X, \mu)$;*

solid *If $f \in L^{p(\cdot)}(X, \mu)$, $g \in L^0(X, \mu)$ (where $L^0(X, \mu)$ stands for the space of all real-valued, μ-measurable functions on X) and $0 \leqslant |g| \leqslant |f|$ μ-almost everywhere, then $g \in L^{p(\cdot)}(X, \mu)$ and $\|g\|_{L^{p(\cdot)}(X,\mu)} \leqslant \|f\|_{L^{p(\cdot)}(X,\mu)}$;*

Fatou's lemma *If $f_k \to f$ μ-almost everywhere, then we have the following: $\|f\|_{L^{p(\cdot)}(X,\mu)} \leqslant \liminf_{k\to\infty} \|f_k\|_{L^{p(\cdot)}(X,\mu)}$;*

Fatou property *If $|f_k| \nearrow |f|$ μ-almost everywhere with $f_k \in L^{p(\cdot)}(X, \mu)$ and $\sup_k \|f_k\|_{L^{p(\cdot)}(X,\mu)} < \infty$, then $f \in L^{p(\cdot)}(X, \mu)$ and $\|f_k\|_{L^{p(\cdot)}(X,\mu)} \nearrow \|f\|_{L^{p(\cdot)}(X,\mu)}$.*

3. Multiplication operators

Definition 3.1. Let $F(X)$ be a function space on a non-empty set X. Let $u : X \to \mathbb{C}$ be a function such that $u \cdot f \in F(X)$ whenever $f \in F(X)$. Then the aplication $f \mapsto u \cdot f$ on $F(X)$ is denoted by M_u. In case $F(X)$ is a topological space and M_u is continuous we call it a *multiplication operator* induced by u.

Multiplication operators generalize the notion of operator given by a diagonal matrix. More precisely, one of the results of operator theory is a spectral theorem, which states that every self-adjoint operator on a Hilbert space is unitarily equivalent to a multiplication operator on an L^2 space.

Consider the Hilbert space $X = L^2[-1, 3]$ of complex-valued square integrable functions on the interval $[-1, 3]$. Define the operator

$$M_u(x) = u(x)x^2,$$

for any function $u \in X$. This will be a self-adjoint bounded linear operator with norm 9. Its spectrum will be the interval $[0, 9]$ (the range of the function $x \to x^2$ defined on $[-1, 3]$). Indeed, for any complex number λ, the operator $M_u - \lambda$ is given by

$$(M_u - \lambda)(x) = (x^2 - \lambda)u(x).$$

It is invertible if and only if λ is not in $[0, 9]$, and then its inverse is

$$(M_u - \lambda)^{-1}(x) = \frac{u(x)}{x^2 - \lambda}.$$

which is another multiplication operator.

For a systematic study of the multiplication operators on different spaces we refer, e.g., to [3, 7, 9, 26].

***Remark* 3.2.** In general, the multiplication operators on measure spaces are not 1–1. Indeed, let (X, Σ, μ) be a measure space and

$$A = X\backslash\text{supp } (u) = \{x \in X : u(x) = 0\}.$$

If $\mu(A) \neq 0$ and $f = \chi_A$ then for any $x \in X$ we have $f(x)u(x) = 0$ which implies that $M_u(f) = 0$, therefore $\ker(M_u) \neq \{0\}$ and hence M_u is not 1–1.

If, on the contrary, M_u is 1–1, then $\mu(X\backslash\text{supp}(u)) = 0$. On the other hand, if $\mu(X\backslash\text{supp}(u)) = 0$ and μ is a complete measure, then $M_u(f) = 0$ implies $f(x)u(x) = 0 \ \forall\, x \in X$, then $\{x \in X : f(x) \neq 0\} \subseteq X\backslash\text{supp}(u)$ and so $f = 0$ μ-a.e. on X.

Hence, if $\mu(X\backslash\text{supp}(u)) = 0$ and μ is a complete measure, then M_u is 1–1.

Proposition 3.3. *M_u is 1–1 on $Y = L^{p(\cdot)}(\text{supp } (u))$.*

***Proof*.** Let $Y = L^{p(\cdot)}(\text{supp } (u)) = \{f\chi_{\text{supp } (u)} : f \in L^{p(\cdot)}(X, \mu)\}$. Indeed, if $M_u(\tilde{f}) = 0$ with $\tilde{f} = f\chi_{\text{supp } (u)} \in Y$, then $f(x)\chi_{\text{supp } (u)}(x)u(x) = 0$ for all $x \in X$, and so

$$\begin{aligned} &f(x)u(x) = 0 \quad \forall\, x \in \text{supp } (u), \\ \Rightarrow\ &f(x) = 0 \quad \forall\, x \in \text{supp } (u), \\ \Rightarrow\ &f(x)\chi_{\text{supp } (u)}(x) = 0 \quad \forall\, x \in X. \end{aligned}$$

Then $\tilde{f} = 0$ and the proof is complete. ☑

In this section we want to characterize the boundedness and compactness of the multiplication operator M_u in variable Lebesgue space $L^{p(\cdot)}(X,\mu)$ in terms of the boundedness and invertibility of the real-valued measurable u function.

3.1. Boundedness results

In the next theorems we will obtain necessary and sufficient conditions related with boundedness and invertibility of the multiplication operator in the framework of variable Lebesgue spaces.

Theorem 3.4. *The linear transformation $M_u : L^{p(\cdot)}(X,\mu) \to L^{p(\cdot)}(X,\mu)$ is bounded if and only if u is essentially bounded. Moreover*

$$\|M_u\|_{L^{p(\cdot)}(X,\mu)\to L^{p(\cdot)}(X,\mu)} = \|u\|_{L^\infty(X,\mu)}.$$

Proof. Letting $u \in L^\infty(X,\mu)$ we then have the pointwise estimate $|(u\cdot f)(x)| \leqslant \|u\|_{L^\infty(X,\mu)}|f(x)|$. Using the fact that $L^{p(\cdot)}(X,\mu)$ is solid, we have

$$\|M_u f\|_{L^{p(\cdot)}(X,\mu)} \leqslant \|u\|_{L^\infty(X,\mu)}\|f\|_{L^{p(\cdot)}(X,\mu)}. \tag{2}$$

On the other hand, suppose that M_u is a bounded operator. If u is not essentially bounded, then for every $n \in \mathbb{N}$, the set $U_n = \{x \in X : |u(x)| > n\}$ has positive measure. Since the measure is σ-finite, there exists a measurable subset of U_n with finite positive measure, denote it by $\widetilde{U}_n$. Then $\chi_{\widetilde{U}_n} \in L^{p(\cdot)}(X,\mu)$ and since $L^{p(\cdot)}(X,\mu)$ is solid and $(|u|\chi_{\widetilde{U}_n})(x) \geqslant n\chi_{\widetilde{U}_n}(x)$ we obtain $\|M_u\chi_{\widetilde{U}_n}\|_{L^{p(\cdot)}(X,\mu)} \geqslant n\|\chi_{\widetilde{U}_n}\|_{L^{p(\cdot)}(X,\mu)}$. This contradicts the supposition that M_u is bounded, therefore u is essentially bounded.

To evaluate the norm of the operator M_u we proceed as follows. For a fixed $\varepsilon > 0$, let us define $U = \{x \in X : |u(x)| \geqslant \|u\|_{L^\infty(X,\mu)} - \varepsilon\}$ which has positive measure. Since the variable Lebesgue space is solid, we have

$$\left\| \frac{(\|u\|_{L^\infty(X,\mu)} - \varepsilon)\chi_U}{\|M_u\chi_U\|_{L^{p(\cdot)}(X,\mu)}} \right\|_{L^{p(\cdot)}(X,\mu)} \leqslant 1$$

which yields $\|M_u\|_{L^{p(\cdot)}(X,\mu)\to L^{p(\cdot)}(X,\mu)} \geqslant \|u\|_{L^\infty(X,\mu)} - \varepsilon$. Since $\varepsilon > 0$ is arbitrary and taking (2) into account, we prove that the norm is equal to $\|u\|_\infty$. ☑

Theorem 3.5. *Let (X,Σ,μ) be a finite measure space. The set of all multiplication operators on $L^{p(\cdot)}(X,\mu)$ is a maximal Abelian subalgebra of $\mathcal{B}(L^{p(\cdot)}(X,\mu))$, the algebra of all bounded operators on $L^{p(\cdot)}(X,\mu)$.*

Proof. Let

$$\mathscr{H} = \{M_u : u \in L^\infty(X,\mu)\} \tag{3}$$

and consider the operator product

$$M_u \cdot M_v = M_{uv},$$

where $M_u, M_v \in \mathscr{H}$. Let us check that $\mathscr{H}$ is a Banach algebra. Let $u, v \in L^\infty(X,\mu)$, then by the pointwise estimates $|u| \leqslant \|u\|_{L^\infty(X,\mu)}$, $|v| \leqslant \|v\|_{L^\infty(X,\mu)}$ we have

$$\|uv\|_{L^\infty} \leqslant \|u\|_{L^\infty(X,\mu)}\|v\|_{L^\infty(X,\mu)}$$

which implies that product is an inner operation, moreover, the usual function product is associative, commutative and distributive with respect to the sum and scalar product, thus $\mathscr{H}$ is a subalgebra of $\mathcal{B}(L^{p(\cdot)}(X,\mu))$.

We will now prove that it is a maximal Abelian subalgebra. Consider the unit function $e : X \to \mathbb{R}$ given by $x \mapsto 1$ for all $x \in X$. Let $N \in \mathcal{B}(L^{p(\cdot)}(X,\mu))$ be an operator which commutes with $\mathscr{H}$ and let χ_E be the indicator function of a measurable set E. Then

$$N(\chi_E) = N[M_{\chi_E}(e)] = M_{\chi_E}[N(e)] = \chi_E \cdot N(e) = N(e) \cdot \chi_E = M_w(\chi_E)$$

where $w := N(e)$. Similarly

$$N(s) = M_w(s), \tag{4}$$

for any simple function.

Now, let us check that $w \in L^\infty(X,\mu)$. By way of contradiction, we assume that $w \notin L^\infty(X,\mu)$, then the set

$$W_n = \{x \in X : |w(x)| > n\},$$

has positive measure for each $n \in \mathbb{N}$. Note that we have the pointwise estimate

$$M_w(\chi_{W_n})(x) = (w\chi_{W_n})(x) \geqslant n\chi_{W_n}(x) \tag{5}$$

for all $x \in X$. Using the fact that $L^{p(\cdot)}(X,\mu)$ is solid and the pointwise estimate (5) we obtain

$$\|M_w(\chi_{W_n})\|_{L^{p(\cdot)}(X,\mu)} = \|w(\chi_{W_n})\|_{L^{p(\cdot)}(X,\mu)} \geqslant n\|\chi_{W_n}\|_{L^{p(\cdot)}(X,\mu)}. \tag{6}$$

Using (4) and (6) we obtain

$$\|N(\chi_{W_n})\|_{L^{p(\cdot)}(X,\mu)} \geqslant n\|\chi_{W_n}\|_{L^{p(\cdot)}(X,\mu)}$$

which contradicts the fact that N is a bounded operator. Therefore $w \in L^\infty(X,\mu)$ and by Theorem 3.4 M_w is bounded.

To obtain the result for all functions in $L^{p(\cdot)}(X,\mu)$ we proceed with a limiting argument. Taking, without loss of generality, a non-negative function

$u \in L^{p(\cdot)}(X,\mu)$, there exists a nondecreasing sequence $\{s_n\}_{s=1}^{\infty}$ of measurable simple functions such that $\lim_{n\to\infty} s_n = u$. Using (4) we have

$$N(u) = N\left(\lim_{n\to\infty} s_n\right) = \lim_{n\to\infty} N(s_n) = \lim_{n\to\infty} M_w(s_n) = M_w\left(\lim_{n\to\infty} s_n\right) = M_w(u),$$

which gives that $N(u) = M_w(u)$ for all $u \in L^{p(\cdot)}(X,\mu)$ and thus $N \in \mathscr{H}$. ☑

Corollary 3.6. *Let (X,Σ,μ) be a finite measure space. The multiplication operator M_u in $L^{p(\cdot)}(X,\mu)$ is invertible if and only if u is invertible in $L^\infty(X,\mu)$.*

Proof. Let M_u be invertible, then there exists $N \in \mathcal{B}(L^{p(\cdot)}(X,\mu))$ such that

$$M_u \cdot N = N \cdot M_u = \mathbb{1} \tag{7}$$

where $\mathbb{1}$ represents the identity operator. Let us check that N commutes with $\mathscr{H}$ (where $\mathscr{H}$ is defined in (3)). Let $M_w \in \mathscr{H}$, then

$$M_w \cdot M_u = M_u \cdot M_w. \tag{8}$$

Applying N to both sides of (8) and using (7) we obtain

$$N\cdot M_w = N\cdot M_w\cdot \mathbb{1} = N\cdot M_w\cdot M_u\cdot N = N\cdot M_u\cdot M_w\cdot N = \mathbb{1}\cdot M_w\cdot N = M_w\cdot N,$$

and thus we conclude that N commutes with $\mathscr{H}$. Then $N \in \mathscr{H}$ by Theorem 3.5 and using again Theorem 3.5 we have that there exists $g \in L^\infty(X,\mu)$ such that $N = M_g$, hence

$$M_u \cdot M_g = M_g \cdot M_u = \mathbb{1}$$

and this implies $ug = gu = 1$ μ-almost everywhere, this means that u is invertible in $L^\infty(X,\mu)$.

On the other hand, assume that u is invertible on $L^\infty(X,\mu)$, that is, $u^{-1} \in L^\infty(X,\mu)$, then

$$M_u \cdot M_{u^{-1}} = M_{u^{-1}} \cdot M_u = M_{u^{-1}u} = M_1 = \mathbb{1}$$

which means that M_u is invertible on $\mathcal{B}(L^{p(\cdot)}(X,\mu))$. ☑

3.2. Compactness results

In the next theorems we will obtain necessary and sufficient conditions related with compactness of the multiplication operator in the framework of variable Lebesgue spaces. We need some definitions for further results, namely

Definition 3.7. For the set X, the real-valued essentially bounded function u and non-negative ε we define the set

$$\mathscr{X}(u,\varepsilon) := \{x \in X : |u(x)| \geqslant \varepsilon\}.$$

With this newly defined set $\mathscr{X}(u,\varepsilon)$ we restrict our space $L^{p(\cdot)}(X,\mu)$, namely

Definition 3.8. We define the space $L^{p(\cdot)}(\mathscr{X}(u,\varepsilon))$ as

$$L^{p(\cdot)}(\mathscr{X}(u,\varepsilon)) := \{f\chi_{\mathscr{X}(u,\varepsilon)} : f \in L^{p(\cdot)}(X,\mu)\}.$$

Lemma 3.9. *The space $L^{p(\cdot)}(\mathscr{X}(u,\varepsilon))$ is a closed invariant subspace of the variable Lebesgue space $L^{p(\cdot)}(X,\mu)$ under M_u.*

Proof. Let $h, s \in L^{p(\cdot)}(\mathscr{X}(u,\varepsilon))$ and $\alpha, \beta \in \mathbb{R}$. Then $h = f\chi_{\mathscr{X}(u,\varepsilon)}$ and $s = g\chi_{\mathscr{X}(u,\varepsilon)}$, where $f, g \in L^{p(\cdot)}(X,\mu)$, thus

$$\alpha h + \beta s = \alpha\left(f\chi_{\mathscr{X}(u,\varepsilon)}\right) + \beta\left(g\chi_{\mathscr{X}(u,\varepsilon)}\right) = (\alpha f + \beta g)\chi_{\mathscr{X}(u,\varepsilon)} \in L^{p(\cdot)}(\mathscr{X}(u,\varepsilon))$$

yielding that $L^{p(\cdot)}(\mathscr{X}(u,\varepsilon))$ is a subspace of $L^{p(\cdot)}(X,\mu)$.

We have that

$$M_u h = uh = uf\chi_{\mathscr{X}(u,\varepsilon)} = (uf)\chi_{\mathscr{X}(u,\varepsilon)}$$

where $uf \in L^{p(\cdot)}(X,\mu)$. Therefore, $M_u h \in L^{p(\cdot)}(\mathscr{X}(u,\varepsilon))$, which means that $L^{p(\cdot)}(\mathscr{X}(u,\varepsilon))$ is an invariant subspace of $L^{p(\cdot)}(X,\mu)$ under M_u.

Now, let us show that $L^{p(\cdot)}(\mathscr{X}(u,\varepsilon))$ is a closed set. Indeed, let us take a function $\mathfrak{g} \in \overline{L^{p(\cdot)}(\mathscr{X}(u,\varepsilon))}$ which also belongs to $L^{p(\cdot)}(X,\mu)$ since it is a Banach space. Then there exists a sequence $\{\mathfrak{g}_n\}_{n=1}^{\infty}$ in $L^{p(\cdot)}(\mathscr{X}(u,\varepsilon))$ such that $\mathfrak{g}_n \to \mathfrak{g}$ in $L^{p(\cdot)}(\mathscr{X}(u,\varepsilon))$. It remains just to show that $\mathfrak{g}$ belongs to $L^{p(\cdot)}(\mathscr{X}(u,\varepsilon))$. Note that

$$\mathfrak{g} = \mathfrak{g}\chi_{\mathscr{X}(u,\varepsilon)} + \mathfrak{g}\chi_{(\mathscr{X}(u,\varepsilon))^{\complement}}.$$

Next, we want to show that $\mathfrak{g}\chi_{(\mathscr{X}(u,\varepsilon))^{\complement}} = 0$. In fact, given $\varepsilon_1 > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}\|\mathfrak{g}\chi_{(\mathscr{X}(u,\varepsilon))^{\complement}}\|_{L^{p(\cdot)}(X,\mu)} &= \|(\mathfrak{g} - \mathfrak{g}_{n_0} + \mathfrak{g}_{n_0})\chi_{(\mathscr{X}(u,\varepsilon))^{\complement}}\|_{L^{p(\cdot)}(X,\mu)} \\ &= \|(\mathfrak{g} - \mathfrak{g}_{n_0})\chi_{(\mathscr{X}(u,\varepsilon))^{\complement}}\|_{L^{p(\cdot)}(X,\mu)} \\ &\leqslant \|\mathfrak{g} - \mathfrak{g}_{n_0}\|_{L^{p(\cdot)}(X,\mu)} \\ &< \varepsilon_1\end{aligned}$$

implying that $\mathfrak{g}\chi_{(\mathscr{X}(u,\varepsilon))^{\complement}} = 0$, hence $\mathfrak{g} \in L^{p(\cdot)}(\mathscr{X}(u,\varepsilon))$. ☑

Theorem 3.10. *Let $u \in L^{\infty}(X,\mu)$. Then the operator M_u is compact if and only if the space $L^{p(\cdot)}(\mathscr{X}(u,\varepsilon))$ is finite dimensional for each $\varepsilon > 0$.*

Proof. For each $f \in L^{p(\cdot)}(X,\mu)$, we have the pointwise estimate

$$|uf\chi_{\mathscr{X}(u,\varepsilon)}(x)| \geqslant \varepsilon|f|\chi_{\mathscr{X}(u,\varepsilon)}(x).$$

Since $L^{p(\cdot)}(X,\mu)$ is solid, we obtain

$$\|M_u f\chi_{\mathscr{X}(u,\varepsilon)}\|_{L^{p(\cdot)}(X,\mu)} \geqslant \varepsilon\|f\chi_{\mathscr{X}(u,\varepsilon)}\|_{L^{p(\cdot)}(X,\mu)}. \tag{9}$$

From Lemma 3.9 we have that $L^{p(\cdot)}(\mathscr{X}(u,\varepsilon))$ is a closed invariant subspace of $L^{p(\cdot)}(X,\mu)$ under M_u which implies that

$$M_u\big|_{L^{p(\cdot)}(\mathscr{X}(u,\varepsilon))}$$

is a compact operator. Then by (9) the operator $M_u\big|_{L^{p(\cdot)}(\mathscr{X}(u,\varepsilon))}$ has closed range in $L^{p(\cdot)}(\mathscr{X}(u,\varepsilon))$ and it is invertible, being compact, $L^{p(\cdot)}(\mathscr{X}(u,\varepsilon))$ is finite dimensional.

Conversely, suppose that $L^{p(\cdot)}(\mathscr{X}(u,\varepsilon))$ is finite dimensional for each $\varepsilon > 0$. In particular, for $n \in \mathbb{N}$, $L^{p(\cdot)}(\mathscr{X}(u,1/n))$ is finite dimensional, then for each n, define

$$u_n : X \to \mathbb{R}$$

as

$$u_n(x) = \begin{cases} u(x), & \text{if } |u(x)| \geqslant \frac{1}{n}, \\ 0, & \text{if } |u(x)| < \frac{1}{n}. \end{cases}$$

Then we find that

$$|M_{u_n}f - M_u f| = |(u_n - u)\cdot f| \leqslant \|u_n - u\|_\infty |f|,$$

yielding, since $L^{p(\cdot)}(X,\mu)$ is solid,

$$\|M_{u_n}f - M_u f\|_{L^{p(\cdot)}(X,\mu)} \leqslant \|u_n - u\|_\infty \|f\|_{L^{p(\cdot)}(X,\mu)}.$$

Consequently

$$\|M_{u_n}f - M_u f\|_{L^{p(\cdot)}(X,\mu)} \leqslant \frac{1}{n}\|f\|_{L^{p(\cdot)}(X,\mu)}$$

which implies that M_{u_n} converges to M_u uniformly. Therefore M_u is compact since it is the limit of operators with finite range. ☑

Theorem 3.11. *Let $u \in L^\infty(X,\mu)$. Then M_u has closed range if and only if there exists a $\delta > 0$ such that $|u(x)| \geqslant \delta$ μ-almost everywhere on* supp (u).

Proof. If there exists $\delta > 0$ such that $|u(x)| \geqslant \delta$ μ-almost everywhere on supp (u), then for all $f \in L^{p(\cdot)}(X,\mu)$ we have

$$\|M_u f\chi_{\text{supp }(u)}\|_{L^{p(\cdot)}(X,\mu)} \geqslant \delta\|f\chi_{\text{supp }(u)}\|_{L^{p(\cdot)}(X,\mu)}$$

and hence M_u has closed range.

Conversely, by [19, Lemma VI.6.1], if M_u has closed range, then there exists an $\varepsilon > 0$ such that

$$\|M_u f\|_{L^{p(\cdot)}(X,\mu)} \geqslant \varepsilon \|f\|_{L^{p(\cdot)}(X,\mu)}$$

for all $f \in L^{p(\cdot)}(\text{supp }(u))$, where

$$L^{p(\cdot)}(\text{supp }(u)) := \left\{ f\chi_{\text{supp }(u)} : f \in L^{p(\cdot)}(X,\mu) \right\}.$$

Let $E = \{x \in \text{supp }(u) : |u(x)| < \varepsilon/2\}$. If $\mu(E) > 0$, then by the σ-finiteness of measure we can find a measurable set $F \subseteq E$ such that $\chi_F \in L^{p(\cdot)}(\text{supp }(u))$, implying that

$$\|M_u \chi_F\|_{L^{p(\cdot)}(X,\mu)} \leqslant \frac{\varepsilon}{2} \|\chi_F\|_{L^{p(\cdot)}(X,\mu)}$$

which is a contradiction. Therefore $\mu(E) = 0$, and this completes the proof. ☑

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