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Voronovskaya Type Asymptotic Expansions for Error Function Based Quasi-Interpolation Neural Network Operators

Expansiones asintóticas de tipo Voronovskaya para funciones de error basadas en cuasi-interpolación de operadores de redes neuronales

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Abstract. Here we examine the quasi-interpolation error function based neural network operators of one hidden layer. Based on fractional calculus theory we derive a fractional Voronovskaya type asymptotic expansion for the error of approximation of these operators to the unit operator, as we are studying the univariate case. We treat also analogously the multivariate case.

Key words and phrases. Neural Network Fractional Approximation, Multivariate Neural Network Approximation, Voronovskaya Asymptotic Expansions, Fractional derivative, Error function.

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Resumen. Aquí se examinan funciones de error basadas en cuasi-interpolación de operadores de redes neuronales de una capa oculta. Basado en teoría de cálculo fraccional se deriva una expansión de asintótica de tipo Voronovskaya para el error de aproximación de estos operadores al operador unitario, así como el caso univariado. También se trata análogamente el caso multivariado.

Palabras y frases clave. Aproximación fraccional de redes neuronales, expansión asintótica de Voronovskaya, derivada fraccional, función error.

1. Background

We consider here the (Gauss) error special function ([1, 17])

$$\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt, \qquad x\in\mathbb{R}, \tag{1}$$

which is a sigmoidal type function and a strictly increasing function.

It has the basic properties

$$\begin{aligned}\operatorname{erf}(0)&=0,\\ \operatorname{erf}(-x)&=-\operatorname{erf}(x),\\ \operatorname{erf}(+\infty)&=1,\\ \operatorname{erf}(-\infty)&=-1.\end{aligned}$$

We consider the activation function ([16])

$$\chi(x)=\frac{1}{4}\big(\operatorname{erf}(x+1)-\operatorname{erf}(x-1)\big), \qquad \text{any } x\in\mathbb{R}, \tag{2}$$

which is an even positive function.

Next we follow [16] on χ. We got there $\chi(0)\simeq 0.4215$, and that χ is strictly decreasing on $[0,\infty)$ and strictly increasing on $(-\infty,0]$, and the x-axis is the horizontal asymptote on χ, i.e. χ is a bell symmetric function.

Theorem 1.1. *([16]) We have that*

$$\sum_{i=-\infty}^{\infty}\chi(x-i)=1, \qquad \textit{all} \quad x\in\mathbb{R}, \tag{3}$$

$$\sum_{i=-\infty}^{\infty}\chi(nx-i)=1, \qquad \textit{all} \quad x\in\mathbb{R},\ n\in\mathbb{N}, \tag{4}$$

and

$$\int_{-\infty}^{\infty}\chi(x)dx=1, \tag{5}$$

that is $\chi(x)$ is a density function on $\mathbb{R}$.

We need the important

Theorem 1.2. *([16]) Let $0<\alpha<1$, and $n\in\mathbb{N}$ with $n^{1-\alpha}\geq 3$. It holds*

$$\sum_{\begin{cases} k=-\infty \\ :|nx-k|\geq n^{1-\alpha}\end{cases}}^{\infty}\chi(nx-k)<\frac{1}{2\sqrt{\pi}\big(n^{1-\alpha}-2\big)e^{(n^{1-\alpha}-2)^2}}. \tag{6}$$

Denote by $\lfloor \bullet \rfloor$ the integral part of the number and by $\lceil \bullet \rceil$ the ceiling of the number.

Theorem 1.3. *([16]) Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx-k)} < \frac{1}{\chi(1)} \simeq 4.019, \qquad \forall x \in [a,b]. \tag{7}$$

Also from [16] we get

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx-k) \neq 1, \tag{8}$$

at least for some $x \in [a,b]$.

For large enough n we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$. In general it holds by (4) that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx-k) \leq 1. \tag{9}$$

We need the univariate neural network operator

Definition 1.4. ([16]) Let $f \in C\big([a,b]\big)$, $n \in \mathbb{N}$. We set

$$A_n(f,x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\big(\frac{k}{n}\big)\chi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx-k)}, \qquad \forall x \in [a,b], \tag{10}$$

A_n is a univariate neural network operator.

We mention from [15] the following:

We define

$$Z(x_1,\ldots,x_N) := Z(x) := \prod_{i=1}^{N} \chi(x_i), \quad x=(x_1,\ldots,x_N) \in \mathbb{R}^N, \quad N \in \mathbb{N}. \tag{11}$$

It has the properties:

(i) $Z(x) > 0$, $\forall x \in \mathbb{R}^N$,

(ii)

$$\sum_{k=-\infty}^{\infty} Z(x-k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} Z(x_1-k_1,\dots,x_N-k_N) = 1, \quad (12)$$

where $k := (k_1,\dots,k_n) \in \mathbb{Z}^N$, $\forall x \in \mathbb{R}^N$, hence

(iii)

$$\sum_{k=-\infty}^{\infty} Z(nx-k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} Z(nx_1-k_1,\dots,nx_N-k_N) = 1, \quad (13)$$

$\forall x \in \mathbb{R}^N$; $n \in \mathbb{N}$, and

(iv)

$$\int_{\mathbb{R}^N} Z(x)\,dx = 1, \quad (14)$$

that is Z is a multivariate density function.

Here $\|x\|_\infty := \max\{|x_1|,\dots,|x_N|\}$, $x \in \mathbb{R}^N$, also set $\infty := (\infty,\dots,\infty)$, $-\infty := (-\infty,\dots,-\infty)$ upon the multivariate context, and

$$\begin{aligned} \lceil na \rceil &:= \big(\lceil na_1 \rceil,\dots,\lceil na_N \rceil\big), \\ \lfloor nb \rfloor &:= \big(\lfloor nb_1 \rfloor,\dots,\lfloor nb_N \rfloor\big), \end{aligned} \quad (15)$$

where $a := (a_1,\dots,a_N)$, $b := (b_1,\dots,b_N)$.

We obviously see that

$$\begin{aligned} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) &= \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(\prod_{i=1}^{N} \chi(nx_i-k_i) \right) \\ &= \sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \cdots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} \left(\prod_{i=1}^{N} \chi(nx_i-k_i) \right) \\ &= \prod_{i=1}^{N} \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \chi(nx_i-k_i) \right). \end{aligned} \quad (16)$$

For $0 < \beta < 1$ and $n \in \mathbb{N}$, a fixed $x \in \mathbb{R}^N$, we have that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx-k) = \sum_{\begin{cases} k=\lceil na \rceil \\ \left\| \frac{k}{n}-x \right\|_\infty \le \frac{1}{n^\beta} \end{cases}}^{\lfloor nb \rfloor} \chi(nx-k) + \sum_{\begin{cases} k=\lceil na \rceil \\ \left\| \frac{k}{n}-x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\lfloor nb \rfloor} \chi(nx-k). \quad (17)$$

In the last two sums the counting is over disjoint vector sets of k's, because the condition $\|\frac{k}{n}-x\|_\infty > \frac{1}{n^\beta}$ implies that there exists at least one $|\frac{k_r}{n}-x_r| > \frac{1}{n^\beta}$, where $r \in \{1, \dots, N\}$.

From [15] we need

(v)

$$\sum_{\begin{cases} k=\lceil na \rceil \\ \left\| \frac{k}{n}-x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\lfloor nb \rfloor} Z(nx-k) \le \frac{1}{2\sqrt{\pi}(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^2}}, \quad (18)$$

$0 < \beta < 1$, $n \in \mathbb{N}$; $n^{1-\beta} \ge 3$, $x \in \left(\prod_{i=1}^N [a_i, b_i] \right)$,

(vi)

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)} < \frac{1}{\left(\chi(1)\right)^N} \simeq (4.019)^N, \quad (19)$$

$\forall x \in \left(\prod_{i=1}^N [a_i, b_i] \right)$, $n \in \mathbb{N}$, and

(vii)

$$\sum_{\begin{cases} k=-\infty \\ \left\| \frac{k}{n}-x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\infty} Z(nx-k) \le \frac{1}{2\sqrt{\pi}(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^2}}, \quad (20)$$

$0 < \beta < 1$, $n \in \mathbb{N}$; $n^{1-\beta} \ge 3$, $x \in \left(\prod_{i=1}^N [a_i, b_i] \right)$.

Also we get that

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) \neq 1, \quad (21)$$

for at least some $x \in \left(\prod_{i=1}^N [a_i, b_i] \right)$.

Let $f \in C\left(\prod_{i=1}^{N}[a_i, b_i]\right)$ and $n \in \mathbb{N}$ such that $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, \ldots, N$.

We mention from [15] the multivariate positive linear neural network operator $\left(x := (x_1, \ldots, x_N) \in \left(\prod_{i=1}^{N}[a_i, b_i]\right)\right)$

$$H_n(f, x) := H_n(f, x_1, \ldots, x_N) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) Z(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)} \tag{22}$$

$$:= \frac{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} \cdots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n}, \ldots, \frac{k_N}{n}\right) \left(\prod_{i=1}^{N} \chi(nx_i - k_i)\right)}{\prod_{i=1}^{N} \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \chi(nx_i - k_i)\right)}.$$

For large enough n we always obtain $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, \ldots, N$. Also $a_i \leq \frac{k_i}{n} \leq b_i$, iff $\lceil na_i \rceil \leq k_i \leq \lfloor nb_i \rfloor$, $i = 1, \ldots, N$.

By $AC^m\left(\prod_{i=1}^{N}[a_i, b_i]\right)$, $m, N \in \mathbb{N}$, we denote the space of functions such that all partial derivatives of order $(m-1)$ of f are coordinatewise absolutely continuous functions, also $f \in C^{m-1}\left(\prod_{i=1}^{N}[a_i, b_i]\right)$.

Let $f \in AC^m\left(\prod_{i=1}^{N}[a_i, b_i]\right)$, $m, N \in \mathbb{N}$. Here f_α denotes a partial derivative of f, $\alpha := (\alpha_1, \ldots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, N$, and $|\alpha| := \sum_{i=1}^{N} \alpha_i = l$, were $l = 0, 1, \ldots, m$. We write also $f_\alpha := \frac{\partial^\alpha f}{\partial x^\alpha}$ and we say it is order l.

We denote

$$||f_\alpha||_{\infty,m}^{\max} := \max_{|\alpha|=m} \left\{||f_\alpha||_\infty\right\}, \tag{23}$$

where $||\bullet||_\infty$ is the supremum norm.

We assume here that $||f_\alpha||_{\infty,m}^{\max} < \infty$.

We need

Definition 1.5. Let $\nu > 0$, $n = \lceil \nu \rceil$ ($\lceil \bullet \rceil$ is the ceiling of the number), $f \in AC^n([a, b])$ (space of functions f with $f^{(n-1)} \in AC([a, b])$, absolutely continuous functions). We call left Caputo fractional derivative (see [21, pp. 49-52]) the function

$$D_{*a}^{\nu} f(x) = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\, dt, \tag{24}$$

$\forall x \in [a, b]$, where Γ is the gamma function $\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1}\, dt$, $\nu > 0$. Notice $D_{*a}^{\nu} f \in L_1([a, b])$ and $D_{*a}^{\nu} f$ exists a.e. on $[a, b]$.

We set $D_{*a}^{0} f(x) = f(x)$, $\forall x \in [a, b]$.

Definition 1.6. (see also [6, 22, 23]). Let $f \in AC^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$. The right Caputo fractional derivative of order $\alpha > 0$ is given by

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (\zeta - x)^{m-\alpha-1} f^{(m)}(\zeta)\, d\zeta, \tag{25}$$

$\forall x \in [a,b]$. We set $D_{b-}^0 f(x) = f(x)$. Notice $D_{b-}^{\alpha} f \in L_1([a,b])$ and $D_{b-}^{\alpha} f$ exists a.e. on $[a,b]$.

Convention 1.7. We assume that

$$D_{*x_0}^{\alpha} f(x) = 0, \qquad \text{for } x < x_0, \tag{26}$$

and

$$D_{x_0-}^{\alpha} f(x) = 0, \qquad \text{for } x > x_0, \tag{27}$$

for all $x, x_0 \in (a,b]$.

We mention

Proposition 1.8. *(by [5]) Let $f \in C^n([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$. Then $D_{*a}^{\nu} f(x)$ is continuous in $x \in [a,b]$.*

Also we have

Proposition 1.9. *(by [5]) Let $f \in C^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then $D_{b-}^{\alpha} f(x)$ is continuous in $x \in [a,b]$.*

Theorem 1.10. *([5]) Let $f \in C^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $x, x_0 \in [a,b]$. Then $D_{*x_0}^{\alpha} f(x)$, $D_{x_0-}^{\alpha} f(x)$ are jointly continuous functions in (x, x_0) from $[a,b]^2$ into $\mathbb{R}$.*

We mention the left Caputo fractional Taylor formula with integral remainder.

Theorem 1.11. *([21, p. 54]) Let $f \in AC^m([a,b])$, $[a,b] \subset \mathbb{R}$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then*

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x - J)^{\alpha-1} D_{*x_0}^{\alpha} f(J)\, dJ, \tag{28}$$

$\forall x \geq x_0$; $x, x_0 \in [a,b]$.

Also we mention the right Caputo fractional Taylor formula.

Theorem 1.12. *([6]) Let $f \in AC^m([a,b])$, $[a,b] \subset \mathbb{R}$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then*

$$f(x) = \sum_{j=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{1}{\Gamma(\alpha)} \int_{x}^{x_0} (J - x)^{\alpha-1} D_{x_0-}^{\alpha} f(J)\, dJ, \tag{29}$$

$\forall x \leq x_0$; $x, x_0 \in [a,b]$.

For more on fractional calculus related to this work see [3, 4, 8].

Next we follow [7, pp. 284-286].

About Taylor formula Multivariate Case and Estimates

Let Q be a compact convex subset of $\mathbb{R}^N$; $N \geq 2$; $z := (z_1, \ldots, z_N)$, $x_0 := (x_{01}, \ldots, x_{0N}) \in Q$.

Let $f : Q \to \mathbb{R}$ be such that all partial derivatives of order $(m-1)$ are coordinatewise absolutely continuous functions, $m \in \mathbb{N}$. Also $f \in C^{m-1}(Q)$. That is $f \in AC^m(Q)$. Each m^{th} order partial derivative is denoted by $f_\alpha := \frac{\partial^\alpha f}{\partial x^\alpha}$, where $\alpha := (\alpha_1, \ldots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, N$ and $|\alpha| := \sum_{i=1}^N \alpha_i = m$. Consider $g_z(t) := f\big(x_0 + t(z - x_0)\big)$, $t \geq 0$. Then

$$g_z^{(j)}(t) = \left[\left(\sum_{i=1}^{N}(z_i - x_{0i})\frac{\partial}{\partial x_i}\right)^j f\right]\big(x_{01} + t(z_1 - x_{01}), \ldots, x_{0N} + t(z_N - x_{0N})\big), \quad (30)$$

for all $j = 0, 1, 2, \ldots, m$.

Example 1.13. Let $m = N = 2$. Then

$$g_z(t) = f\big(x_{01} + t(z_1 - x_{01}), x_{02} + t(z_2 - x_{02})\big), \qquad t \in \mathbb{R},$$

and

$$g_z'(t) = (z_1 - x_{01})\frac{\partial f}{\partial x_1}\big(x_0 + t(z - x_0)\big) + (z_2 - x_{02})\frac{\partial f}{\partial x_2}\big(x_0 + t(z - x_0)\big). \quad (31)$$

Setting

$$(*) = \big(x_{01} + t(z_1 - x_{01}), x_{02} + t(z_2 - x_{02})\big) = \big(x_0 + t(z - x_0)\big),$$

we get

$$g_z''(t) = (z_1 - x_{01})^2\frac{\partial f^2}{\partial x_1^2}(*) + (z_1 - x_{01})(z_2 - x_{02})\frac{\partial f^2}{\partial x_2 \partial x_1}(*) + (z_1 - x_{01})(z_2 - x_{02})\frac{\partial f^2}{\partial x_1 \partial x_2}(*) + (z_2 - x_{02})^2\frac{\partial f^2}{\partial x_2^2}(*). \quad (32)$$

Similarly, we have the general case of $m, N \in \mathbb{N}$ for $g_z^{(m)}(t)$.

We mention the following multivariate Taylor theorem.

Theorem 1.14. *([7]) Under the above assumptions we have*

$$f(z_1, \ldots, z_N) = g_z(1) = \sum_{j=0}^{m-1}\frac{g_z^{(j)}(0)}{j!} + R_m(z, 0), \quad (33)$$

where

$$R_m(z,0) := \int_0^1 \left(\int_0^{t_1} \cdots \left(\int_0^{t_{m-1}} g_z^{(m)}(t_m)\, dt_m \right) \cdots \right) dt_1, \tag{34}$$

or

$$R_m(z,0) = \frac{1}{(m-1)!} \int_0^1 (1-\theta)^{m-1} g_z^{(m)}(\theta)\, d\theta. \tag{35}$$

Notice that $g_z(0) = f(x_0)$.

We make

***Remark* 1.15.** Assume here that

$$\|f_\alpha\|_{\infty,Q,m}^{\max} := \max_{|\alpha|=m} \|f_\alpha\|_{\infty,Q} < \infty.$$

Then

$$\|g_z^{(m)}\|_{\infty,[0,1]} = \left\| \left[\left(\sum_{i=1}^{N} (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^m f \right] \left(x_0 + t(z - x_0) \right) \right\|_{\infty,[0,1]} \leq$$

$$\left(\sum_{i=1}^{N} |z_i - x_{0i}| \right)^m \|f_\alpha\|_{\infty,Q,m}^{\max}, \tag{36}$$

that is

$$\|g_z^{(m)}\|_{\infty,[0,1]} \leq \left(\|z - x_0\|_{l_1} \right)^m \|f_\alpha\|_{\infty,Q,m}^{\max} < \infty. \tag{37}$$

Hence we get by (35) that

$$\left| R_m(z,0) \right| \leq \frac{\|g_z^{(m)}\|_{\infty,[0,1]}}{m!} < \infty \tag{38}$$

and it holds

$$\left| R_m(z,0) \right| \leq \frac{\left(\|z - x_0\|_{l_1} \right)^m}{m!} \|f_\alpha\|_{\infty,Q,m}^{\max}, \tag{39}$$

$\forall z, x_0 \in Q$.

Inequality (39) will be an important tool in proving our multivariate main result.

In this article first we find fractional Voronskaya type asymptotic expansion for $A_n(f,x)$, $x \in [a,b]$, then we find multivariate Voronskaya type asymptotic expansion for $H_n(f,x)$, $x \in \left(\prod_{i=1}^{N} [a_i, b_i] \right)$; $n \in \mathbb{N}$.

Our considered neural networks here are of one hidden layer.

For other neural networks related work, see [2, 9, 12, 10, 11, 14, 13, 18, 19, 20]. For neural networks in general, read [24, 25, 26].

2. Main Results

We present our first univariate main result

Theorem 2.1. *Let $\alpha > 0$, $N \in \mathbb{N}$, $N = \lceil \alpha \rceil$, $f \in AC^N([a,b])$, $0 < \beta < 1$, $x \in [a,b]$, $n \in \mathbb{N}$ large enough and $n^{1-\beta} \geq 3$. Assume that $\|D^\alpha_{x-}f\|_{\infty,[a,x]}$, $\|D^\alpha_{*x}f\|_{\infty,[x,b]} \leq M$, $M > 0$. Then*

$$A_n(f,x) - f(x) = \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} A_n\big((\bullet - x)^j, x\big) + o\left(\frac{1}{n^{\beta(\alpha-\varepsilon)}}\right), \tag{40}$$

where $0 < \varepsilon \leq \alpha$.

If $N = 1$, the sum in (40) collapses.

The last (40) implies that

$$n^{\beta(\alpha-\varepsilon)}\left[A_n(f,x) - f(x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} A_n\big((\bullet - x)^j, x\big)\right] \to 0, \tag{41}$$

as $n \to \infty$, $0 < \varepsilon \leq \alpha$.

When $N = 1$, or $f^{(j)}(x) = 0$, $j = 1, \dots, N-1$, then we derive that

$$n^{\beta(\alpha-\varepsilon)}\big[A_n(f,x) - f(x)\big] \to 0$$

as $n \to \infty$, $0 < \varepsilon \leq \alpha$. Of great interest is the case of $\alpha = \frac{1}{2}$.

Proof. From [21, p. 54; (28)], we get by the left Caputo fractional Taylor formula that

$$f\left(\frac{k}{n}\right) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!}\left(\frac{k}{n} - x\right)^j + \frac{1}{\Gamma(\alpha)} \int_x^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\alpha-1} D^\alpha_{*x} f(J)\, dJ, \tag{42}$$

for all $x \leq \frac{k}{n} \leq b$.

Also from [6, (29)], using the right Caputo fractional Taylor formula we get

$$f\left(\frac{k}{n}\right) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!}\left(\frac{k}{n} - x\right)^j + \frac{1}{\Gamma(\alpha)} \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n}\right)^{\alpha-1} D^\alpha_{x-} f(J)\, dJ, \tag{43}$$

for all $a \leq \frac{k}{n} \leq x$.

We call

$$W(x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx - k). \tag{44}$$

Hence we have

$$\frac{f\left(\frac{k}{n}\right)\chi(nx-k)}{W(x)}=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\frac{\chi(nx-k)}{W(x)}\left(\frac{k}{n}-x\right)^{j}+\frac{\chi(nx-k)}{W(x)\Gamma(\alpha)}\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}D_{*x}^{\alpha}f(J)\,dJ, \quad (45)$$

all $x\leq\frac{k}{n}\leq b$, iff $\lceil nx\rceil\leq k\leq\lfloor nb\rfloor$, and

$$\frac{f\left(\frac{k}{n}\right)\chi(nx-k)}{W(x)}=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\frac{\chi(nx-k)}{W(x)}\left(\frac{k}{n}-x\right)^{j}+\frac{\chi(nx-k)}{W(x)\Gamma(\alpha)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}D_{x-}^{\alpha}f(J)\,dJ, \quad (46)$$

for all $a\leq\frac{k}{n}\leq x$, iff $\lceil na\rceil\leq k\leq\lfloor nx\rfloor$.

We have that $\lceil nx\rceil\leq\lfloor nx\rfloor+1$.

Therefore it holds

$$\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\frac{f\left(\frac{k}{n}\right)\chi(nx-k)}{W(x)}=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\frac{\chi(nx-k)\left(\frac{k}{n}-x\right)^{j}}{W(x)}+\frac{1}{\Gamma(\alpha)}\left(\frac{\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\chi(nx-k)}{W(x)}\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}D_{*x}^{\alpha}f(J)\,dJ\right), \quad (47)$$

and

$$\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}f\left(\frac{k}{n}\right)\frac{\chi(nx-k)}{W(x)}=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\frac{\chi(nx-k)}{W(x)}\left(\frac{k}{n}-x\right)^{j}+\frac{1}{\Gamma(\alpha)}\left(\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\frac{\chi(nx-k)}{W(x)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}D_{x-}^{\alpha}f(J)\,dJ\right). \quad (48)$$

Adding the last two equalities (47) and (48) we obtain

$$A_n(f,x) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \frac{\chi(nx-k)}{W(x)}$$
$$= \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \frac{\chi(nx-k)}{W(x)} \left(\frac{k}{n} - x\right)^j +$$
$$\frac{1}{\Gamma(\alpha) W(x)} \left\{ \sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \chi(nx-k) \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n}\right)^{\alpha-1} D_{x-}^{\alpha} f(J)\, dJ + \right.$$
$$\left. \sum_{k=\lfloor nx \rfloor + 1}^{\lfloor nb \rfloor} \chi(nx-k) \int_{x}^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\alpha-1} \left(D_{*x}^{\alpha} f(J)\right) dJ \right\}. \quad (49)$$

So we have derived

$$\theta(x) := A_n(f,x) - f(x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} A_n\left((\bullet - x)^j\right)(x) = \theta_n^*(x), \quad (50)$$

where

$$\theta_n^*(x) := \frac{1}{\Gamma(\alpha) W(x)} \left\{ \sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \chi(nx-k) \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n}\right)^{\alpha-1} D_{x-}^{\alpha} f(J)\, dJ + \right.$$
$$\left. \sum_{k=\lfloor nx \rfloor + 1}^{\lfloor nb \rfloor} \chi(nx-k) \int_{x}^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\alpha-1} D_{*x}^{\alpha} f(J)\, dJ \right\}. \quad (51)$$

We set

$$\theta_{1n}^*(x) := \frac{1}{\Gamma(\alpha)} \left(\frac{\sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \chi(nx-k)}{W(x)} \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n}\right)^{\alpha-1} D_{x-}^{\alpha} f(J)\, dJ \right), \quad (52)$$

and

$$\theta_{2n}^* := \frac{1}{\Gamma(\alpha)} \left(\frac{\sum_{k=\lfloor nx \rfloor + 1}^{\lfloor nb \rfloor} \chi(nx-k)}{W(x)} \int_{x}^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\alpha-1} D_{*x}^{\alpha} f(J)\, dJ \right), \quad (53)$$

i.e.,

$$\theta_n^*(x) = \theta_{1n}^*(x) + \theta_{2n}^*(x). \quad (54)$$

We assume $b - a > \frac{1}{n^\beta}$, $0 < \beta < 1$, which is always the case for large enough $n \in \mathbb{N}$, that is when $n > \left\lceil (b-a)^{-\frac{1}{\beta}} \right\rceil$. It is always true that either $\left|\frac{k}{n} - x\right| \leq \frac{1}{n^\beta}$ or $\left|\frac{k}{n} - x\right| > \frac{1}{n^\beta}$.

For $k = \lceil na \rceil, \dots, \lfloor nx \rfloor$, we consider

$$\gamma_{1k} := \left| \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n} \right)^{\alpha - 1} D_{x-}^{\alpha} f(J) \, dJ \right| \\ \leq \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n} \right)^{\alpha - 1} \left| D_{x-}^{\alpha} f(J) \right| dJ \quad (55)$$

$$\leq \| D_{x-}^{\alpha} f \|_{\infty, [a,x]} \frac{\left(x - \frac{\kappa}{n} \right)^{\alpha}}{\alpha} \leq \| D_{x-}^{\alpha} f \|_{\infty, [a,x]} \frac{(x-a)^{\alpha}}{\alpha}. \quad (56)$$

That is

$$\gamma_{1k} \leq \| D_{x-}^{\alpha} f \|_{\infty, [a,x]} \frac{(x-a)^{\alpha}}{\alpha}, \quad (57)$$

for $k = \lceil na \rceil, \dots, \lfloor nx \rfloor$.

Also we have in case of $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\beta}}$ that

$$\gamma_{1k} \leq \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n} \right)^{\alpha - 1} \left| D_{x-}^{\alpha} f(J) \right| dJ \\ \leq \left\| D_{x-}^{\alpha} f \right\|_{\infty, [a,x]} \frac{\left(x - \frac{\kappa}{n} \right)^{\alpha}}{\alpha} \leq \left\| D_{x-}^{\alpha} f \right\|_{\infty, [a,x]} \frac{1}{n^{\alpha \beta} \alpha}. \quad (58)$$

So that, when $\left(x - \frac{k}{n} \right) \leq \frac{1}{n^{\beta}}$, we get

$$\gamma_{1k} \leq \| D_{x-}^{\alpha} f \|_{\infty, [a,x]} \frac{1}{\alpha n^{a \beta}}. \quad (59)$$

Therefore

$$\left| \theta_{1n}^{*}(x) \right| \leq \frac{1}{\Gamma(\alpha)} \left(\frac{\sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \chi(nx-k)}{W(x)} \gamma_{1k} \right) \\ = \frac{1}{\Gamma(\alpha)} \left\{ \frac{\sum_{\left\{ \substack{k=\lceil na \rceil \\ : \left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\beta}}} \right.}^{\lfloor nx \rfloor} \chi(nx-k)}{W(x)} \gamma_{1k} + \frac{\sum_{\left\{ \substack{k=\lceil na \rceil \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^{\beta}}} \right.}^{\lfloor nx \rfloor} \chi(nx-k)}{W(x)} \gamma_{1k} \right\}$$

$$\leq \frac{1}{\Gamma(\alpha)} \left\{ \left(\frac{\sum_{\substack{k=\lceil na \rceil \\ \left\{ : \left| \frac{k}{n} - x \right| \leq \frac{1}{n^\beta} \right.}}^{\lfloor nx \rfloor} \chi(nx-k)}{W(x)} \right) \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} \frac{1}{\alpha n^{\alpha\beta}} + \right.$$

$$\left. \frac{1}{W(x)} \left(\sum_{\substack{k=\lceil na \rceil \\ \left\{ : \left| \frac{k}{n} - x \right| > \frac{1}{n^\beta} \right.}}^{\lfloor nx \rfloor} \chi(nx-k) \right) \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} \frac{(x-a)^\alpha}{\alpha} \right\}$$

$$\overset{\text{(by (6),(7))}}{\leq} \frac{\left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]}}{\Gamma(\alpha+1)} \left\{ \frac{1}{n^{\alpha\beta}} + \frac{4.019}{2\sqrt{\pi}(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^2}} (x-a)^\alpha \right\}. \quad (60)$$

Therefore we proved

$$\left| \theta_{1n}^{*}(x) \right| \leq \frac{\| D_{x-}^{\alpha} f \|_{\infty,[a,x]}}{\Gamma(\alpha+1)} \left\{ \frac{1}{n^{\alpha\beta}} + \frac{2.0095}{\sqrt{\pi}(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^2}} (x-a)^\alpha \right\}. \quad (61)$$

But for large enough $n \in \mathbb{N}$ we get

$$\left| \theta_{1n}^{*}(x) \right| \leq \frac{2 \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]}}{\Gamma(\alpha+1) n^{\alpha\beta}}. \quad (62)$$

Similarly we have

$$\gamma_{2k} := \left| \int_x^{\frac{k}{n}} \left(\frac{k}{n} - J \right)^{\alpha-1} D_{*x}^{\alpha} f(J)\, dJ \right| \leq \int_x^{\frac{k}{n}} \left(\frac{k}{n} - J \right)^{\alpha-1} \left| D_{*x}^{\alpha} f(J) \right| dJ$$

$$\leq \left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]} \frac{\left(\frac{k}{n} - x \right)^\alpha}{\alpha} \leq \left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]} \frac{(b-x)^\alpha}{\alpha}. \quad (63)$$

That is

$$\gamma_{2k} \leq \left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]} \frac{(b-x)^\alpha}{\alpha}, \quad (64)$$

for $k = \lfloor nx \rfloor + 1, \ldots, \lfloor nb \rfloor$.

Also we have in case of $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^\beta}$ that

$$\gamma_{2k} \leq \frac{\left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]}}{\alpha n^{\alpha\beta}}. \quad (65)$$

Consequently it holds

$$\left|\theta_{2n}^{*}(x)\right| \leq \frac{1}{\Gamma(\alpha)}\left(\frac{\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\chi(nx-k)}{W(x)}\gamma_{2k}\right)$$

$$= \frac{1}{\Gamma(\alpha)}\left\{\left(\frac{\sum_{\substack{k=\lfloor nx\rfloor+1\\ :\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\beta}}}}^{\lfloor nb\rfloor}\chi(nx-k)}{W(x)}\right)\frac{\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}}{\alpha n^{\alpha\beta}} + \right.$$

$$\left.\frac{1}{W(x)}\left(\sum_{\substack{k=\lfloor nx\rfloor+1\\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\beta}}}}^{\lfloor nb\rfloor}\chi(nx-k)\right)\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\frac{(b-x)^{\alpha}}{\alpha}\right\}$$

$$\leq \frac{\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}}{\Gamma(\alpha+1)}\left\{\frac{1}{n^{\alpha\beta}} + \frac{2.0095}{\sqrt{\pi}(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^{2}}}(b-x)^{\alpha}\right\}. \quad (66)$$

That is

$$\left|\theta_{2n}^{*}(x)\right| \leq \frac{\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}}{\Gamma(\alpha+1)}\left\{\frac{1}{n^{\alpha\beta}} + \frac{2.0095}{\sqrt{\pi}(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^{2}}}(b-x)^{\alpha}\right\}. \quad (67)$$

But for large enough $n \in \mathbb{N}$ we get

$$\left|\theta_{2n}^{*}(x)\right| \leq \frac{2\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}}{\Gamma(\alpha+1)n^{\alpha\beta}}. \quad (68)$$

Since $\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}$, $\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]} \leq M$, $M > 0$, we derive

$$\left|\theta_{n}^{*}(x)\right| \leq \left|\theta_{1n}^{*}(x)\right| + \left|\theta_{2n}^{*}(x)\right| \overset{\text{(by (62), (68))}}{\leq} \frac{4M}{\Gamma(\alpha+1)n^{\alpha\beta}}. \quad (69)$$

That is for large enough $n \in \mathbb{N}$ we get

$$|\theta(x)| = \left|\theta_{n}^{*}(x)\right| \leq \left(\frac{4M}{\Gamma(\alpha+1)}\right)\left(\frac{1}{n^{\alpha\beta}}\right), \quad (70)$$

resulting to

$$\left|\theta(x)\right| = O\left(\frac{1}{n^{\alpha\beta}}\right) \quad (71)$$

and

$$\left|\theta(x)\right| = o(1). \quad (72)$$

Letting $0 < \varepsilon \leq \alpha$, we derive

$$\frac{\left|\theta(x)\right|}{\left(\frac{1}{n^{\beta(\alpha-\varepsilon)}}\right)} \leq \left(\frac{4M}{\Gamma(\alpha+1)}\right)\left(\frac{1}{n^{\beta\varepsilon}}\right) \to 0 \tag{73}$$

as $n \to \infty$, i.e.

$$\left|\theta(x)\right| = o\left(\frac{1}{n^{\beta(\alpha-\varepsilon)}}\right), \tag{74}$$

proving the claim. ☑

We present our second main result which is a multivariate one.

Theorem 2.2. *Let $0 < \beta < 1$, $x \in \prod_{i=1}^{N}[a_i, b_i]$, $n \in \mathbb{N}$ large enough and $n^{1-\beta} \geq 3$, $f \in AC^m\left(\prod_{i=1}^{N}[a_i, b_i]\right)$, $m, N \in \mathbb{N}$. Assume further that $\left\|f_\alpha\right\|_{\infty,m}^{\max} < \infty$. Then*

$$H_n(f,x) - f(x) = \\ \sum_{j=1}^{m-1}\left(\sum_{|\alpha|=j}\left(\frac{f_\alpha(x)}{\prod_{i=1}^{N}\alpha_i!}\right)H_n\left(\prod_{i=1}^{N}(\bullet - x_i)^{\alpha_i}, x\right)\right) + o\left(\frac{1}{n^{\beta(m-\varepsilon)}}\right), \tag{75}$$

where $0 < \varepsilon \leq m$.

If $m = 1$, the sum in (75) collapses.

It follows from (75) that

$$n^{\beta(m-\varepsilon)}\left[H_n(f,x) - f(x) - \sum_{j=1}^{m-1}\left(\sum_{|\alpha|=j}\left(\frac{f_\alpha(x)}{\prod_{i=1}^{N}\alpha_i!}\right)\right.\right. \\ \left.\left. H_n\left(\prod_{i=1}^{N}(\bullet - x_i)^{\alpha_i}, x\right)\right)\right] \to 0, \quad \text{as} \quad n \to \infty, \quad 0 < \varepsilon \leq m. \tag{76}$$

When $m = 1$, or $f_\alpha(x) = 0$, for $|\alpha| = j$, $j = 1, \ldots, m-1$, then we derive that

$$n^{\beta(m-\varepsilon)}\left[H_n(f,x) - f(x)\right] \to 0$$

as $n \to \infty$, $0 < \varepsilon \leq m$.

Proof. Consider $g_z(t) := f\left(x_0 + t(z - x_0)\right)$, $t \geq 0$; $x_0, z \in \prod_{i=1}^{N}[a_i, b_i]$. Then

$$g_z^{(j)}(t) = \\ \left[\left(\sum_{i=1}^{N}(z_i - x_{0i})\frac{\partial}{\partial x_i}\right)^j f\right]\left(x_{01} + t(z_1 - x_{01}), \ldots, x_{0N} + t(z_N - x_{0N})\right), \tag{77}$$

for all $j = 0, 1, \ldots, m$.

By (33) we have the multivariate Taylor's formula

$$f(z_1, \ldots, z_N) = g_z(1) = \sum_{j=0}^{m-1} \frac{g_z^{(j)}(0)}{j!} + \frac{1}{(m-1)!} \int_0^1 (1-\theta)^{m-1} g_z^{(m)}(\theta)\, d\theta. \quad (78)$$

Notice $g_z(0) = f(x_0)$. Also for $j = 0, 1, \ldots, m-1$, we have

$$g_z^{(j)}(0) = \sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\ \alpha_i \in \mathbb{Z}^+, \\ i=1,\ldots,N,\ |\alpha|:=\sum_{i=1}^N \alpha_i = j}} \left(\frac{j!}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N (z_i - x_{0i})^{\alpha_i} \right) f_\alpha(x_0). \quad (79)$$

Furthermore

$$g_z^{(m)}(\theta) = \sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\ \alpha_i \in \mathbb{Z}^+, \\ i=1,\ldots,N,\ |\alpha|:=\sum_{i=1}^N \alpha_i = m}} \left(\frac{m!}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N (z_i - x_{0i})^{\alpha_i} \right) f_\alpha\big(x_0 + \theta(z - x_0)\big), \quad (80)$$

$0 \le \theta \le 1$.

So we treat $f \in AC^m\Big(\prod_{i=1}^N [a_i, b_i] \Big)$ with $\|f_\alpha\|_{\infty,m}^{\max} < \infty$.

Thus, by (78) we have for $\frac{k}{n}$, $x \in \Big(\prod_{i=1}^N [a_i, b_i] \Big)$ that

$$f\left(\frac{k_1}{n}, \ldots, \frac{k_N}{n} \right) - f(x) = \sum_{j=1}^{m-1} \sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\ \alpha_i \in \mathbb{Z}^+, \\ i=1,\ldots,N,\ |\alpha|:=\sum_{i=1}^N \alpha_i = j}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N \left(\frac{k_i}{n} - x_i \right)^{\alpha_i} \right) f_\alpha(x) + R, \quad (81)$$

where

$$R := m \int_0^1 (1-\theta)^{m-1} \sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\ \alpha_i \in \mathbb{Z}^+, \\ i=1,\ldots,N,\ |\alpha|:=\sum_{i=1}^N \alpha_i = m}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N \left(\frac{k_i}{n} - x_i \right)^{\alpha_i} \right) f_\alpha\left(x + \theta\left(\frac{k}{n} - x \right) \right) d\theta. \quad (82)$$

By (39) we obtain

$$|R| \le \frac{\left(\left\| x - \frac{k}{n} \right\|_{l_1} \right)^m}{m!} \|f_\alpha\|_{\infty,m}^{\max}. \quad (83)$$

Notice here that

$$\left\|\frac{k}{n}-x\right\|_{\infty} \leq \frac{1}{n^{\beta}} \Leftrightarrow \left|\frac{k_i}{n}-x_i\right| \leq \frac{1}{n^{\beta}}, \qquad i=1,\ldots,N. \tag{84}$$

So, if $\left\|\frac{k}{n}-x\right\|_{\infty} \leq \frac{1}{n^{\beta}}$ we get that $\left\|x-\frac{k}{n}\right\|_{l_1} \leq \frac{N}{n^{\beta}}$, and

$$|R| \leq \frac{N^m}{n^{m\beta}m!}\|f_\alpha\|_{\infty,m}^{\max}. \tag{85}$$

Also we see that

$$\left\|x-\frac{k}{n}\right\|_{l_1} = \sum_{i=1}^{N}\left|x_i-\frac{k_i}{n}\right| \leq \sum_{i=1}^{N}(b_i-a_i) = \|b-a\|_{l_1},$$

therefore, in general, it holds

$$|R| \leq \frac{\left(\|b-a\|_{l_1}\right)^m}{m!}\|f_\alpha\|_{\infty,m}^{\max}. \tag{86}$$

Call

$$V(x) := \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k).$$

Hence we have

$$U_n(x) := \frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)R}{V(x)} =$$
$$\frac{\sum_{\begin{cases} k=\lceil na\rceil \\ :\left\|\frac{k}{n}-x\right\|_{\infty}\leq\frac{1}{n^{\beta}}\end{cases}}^{\lfloor nb\rfloor} Z(nx-k)R}{V(x)} + \frac{\sum_{\begin{cases} k=\lceil na\rceil \\ :\left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}\end{cases}}^{\lfloor nb\rfloor} Z(nx-k)R}{V(x)}. \tag{87}$$

Consequently we obtain

$$\left|U_n(x)\right| \leq \left(\frac{\sum_{\substack{k=\lceil na \rceil \\ \left\{ : \left\| \frac{k}{n} - x \right\|_\infty \leq \frac{1}{n^\beta} \right.}}^{\lfloor nb \rfloor} Z(nx-k)}{V(x)} \right) \left(\frac{N^m}{n^{m\beta} m!} \|f_\alpha\|_{\infty,m}^{\max} \right) +$$

$$\frac{1}{V(x)} \left(\sum_{\substack{k=\lceil na \rceil \\ \left\{ : \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \right.}}^{\lfloor nb \rfloor} Z(nx-k) \right) \frac{\left(\|b-a\|_{l_1} \right)^m}{m!} \|f_\alpha\|_{\infty,m}^{\max}$$

$$\overset{\text{(by (19), (18))}}{\leq} \frac{N^m}{n^{m\beta} m!} \|f_\alpha\|_{\infty,m}^{\max} +$$

$$(4.019)^N \frac{1}{2\sqrt{\pi}(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^2}} \frac{\left(\|b-a\|_{l_1} \right)^m}{m!} \|f_\alpha\|_{\infty,m}^{\max}. \quad (88)$$

Therefore we have found

$$|U_n(x)| \leq$$
$$\frac{\|f_\alpha\|_{\infty,m}^{\max}}{m!} \left\{ \frac{N^m}{n^{m\beta}} + (4.019)^N \frac{1}{2\sqrt{\pi}(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^2}} \left(||b-a||_{l_1} \right)^m \right\}. \quad (89)$$

For large enough $n \in \mathbb{N}$ we get

$$\left|U_n(x)\right| \leq \left(\frac{2\|f_\alpha\|_{\infty,m}^{\max} N^m}{m!} \right) \left(\frac{1}{n^{m\beta}} \right). \quad (90)$$

That is

$$|U_n(x)| = O\left(\frac{1}{n^{m\beta}} \right), \quad (91)$$

and

$$|U_n(x)| = o(1). \quad (92)$$

And, letting $0 < \varepsilon \leq m$, we derive

$$\frac{|U_n(x)|}{\left(\frac{1}{n^{\beta(m-\varepsilon)}} \right)} \leq \left(\frac{2\|f_\alpha\|_{\infty,m}^{\max} N^m}{m!} \right) \frac{1}{n^{\beta\varepsilon}} \to 0, \quad (93)$$

as $n \to \infty$, i.e.

$$|U_n(x)| = o\left(\frac{1}{n^{\beta(m-\varepsilon)}} \right). \quad (94)$$

By (81) we observe that

$$\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right)Z(nx-k)}{V(x)} - f(x) =$$

$$\sum_{j=1}^{m-1}\left(\sum_{|\alpha|=j}\left(\frac{f_\alpha(x)}{\prod_{i=1}^{N}\alpha_i!}\right)\right)\frac{\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)\left(\prod_{i=1}^{N}\left(\frac{k_i}{n}-x_i\right)^{\alpha_i}\right)\right)}{V(x)} +$$

$$\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)R}{V(x)}. \quad (95)$$

The last says

$$H_n(f,x) - f(x) - \sum_{j=1}^{m-1}\left(\sum_{|\alpha|=j}\left(\frac{f_\alpha(x)}{\prod_{i=1}^{N}\alpha_i!}\right)H_n\left(\prod_{i=1}^{N}(\cdot - x_i)^{\alpha_i}, x\right)\right) =$$

$$U_n(x). \quad (96)$$

The proof of the theorem is complete. ☑

References

[1] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover Publications, New York, USA, 1972.

[2] G. A. Anastassiou, *Rate of Convergence of Some Neural Network Operators to the Unit-Univariate Case*, J. Math. Anal. Appli. **212** (1997), 237–262.

[3] ______, *Quantitative Approximations*, Chapman & Hall/CRC, New York, USA, 2001.

[4] ______, *Fractional Differentiation Inequalities*, Springer, New York, USA, 2009.

[5] ______, *Fractional Korovkin Theory*, Chaos, Solitons & Fractals **42** (2009), no. 4, 2080–2094.

[6] ______, *On Right Fractional Calculus*, Chaos, Solitons & Fractals **42** (2009), no. 1, 365–376.

[7] ______, *Advanced Inequalities*, World Scientific Publ. Co., New Jersey, USA, 2011.

[8] ______, *Fractional Representation Formulae and Right Fractional Inequalities*, Mathematical and Computer Modelling **54** (2011), no. 11-12, 3098–3115.

[9] ______, *Inteligent Systems: Approximation by Artificial Neural Networks*, Intelligent Systems Reference Library (Heidelberg, Germany), vol. 19, Springer, 2011.

[10] ______, *Multivariate Hyperbolic Tangent Neural Network Approximation*, Computers and Mathematics **61** (2011), 809–821.

[11] ______, *Multivariate Sigmoidal Neural Network Approximation*, Neural Networks (2011).

[12] ______, *Univariate Hyperbolic Tangent Neural Network Approximation*, Mathematics and Computer Modelling **53** (2011), 1111–1132.

[13] ______, *Fractional Neural Network Approximation*, Computers and Mathematics with Applications **64** (2012), 1655–1676.

[14] ______, *Univariate Sigmoidal Neural Network Approximation*, Journal of Computational Analysis and Applications **14** (2012), no. 4, 659–690.

[15] ______, *Multivariate Error Function Based Neural Network Approximations*, submitted, 2014.

[16] ______, *Univariate Error Function Based Neural Network Approximation*, submitted, 2014.

[17] L. C. Andrews, *Special Functions of Mathematics for Engineers*, second ed., Mc Graw-Hill, New York, USA, 1992.

[18] Z. Chen and F. Cao, *The Approximation Operators With Sigmoidal Functions*, Computers and Mathematics with Applications **58** (2009), 758–765.

[19] D. Costarelli and R. Spigler, *Approximation Results for Neural Network Operators Activated by Sigmoidal Functions*, Neural Networks **44** ((2013), 101–106.

[20] ______, *Multivariate Neural Network Operators With Sigmoidal Activation Functions*, Neural Networks **48** (2013), 72–77.

[21] K. Diethelm, *The Analysis of Fractional Differential Equations*, Lecture Notes in Mathematics 2004 (Berlin, Germany), Springer-Verlag, 2010.

[22] A. M. A. El-Sayed and M. Gaber, *On the Finite Caputo and Finite Riesz Derivatives*, Electronic Journal of Theoretical Physics **3** (2006), no. 12, 81–95.

[23] G. S. Frederico and D. F. M. Torres, *Fractional Optimal Control in the Sense of Caputo and the Fractional Noether's Theorem*, International Mathematical Forum **3** (2008), no. 10, 479–493.

[24] S. Haykin, *Neural Networks: A Comprehensive Foundation*, second ed., Prentice Hall, New York, USA, 1998.

[25] W. McCulloch and W. Pitts, *A Logical Calculus of the Ideas Immanent in Nervous Activity*, Bulletin of Mathematical Biophysics **7** (1943), 115–133.

[26] T. M. Mitchell, *Machine Learning*, WCB-McGraw-Hill, New York, USA, 1997.

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