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A Lower Bound for the First Steklov Eigenvalue on a Domain

Una cota inferior para el primer valor propio de Steklov en un dominio

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Abstract. In this paper we provide a lower bound for the first eigenvalue of the Steklov problem in a star-shaped bounded domain in $\mathbb{R}^n$. This result extends to higher dimensions a lower estimate of Kuttler-Sigillito in a two dimensional star-shaped bounded domain.

Key words and phrases. Eigenvalue, Lower bound , The Steklov problem.

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Resumen. En este trabajo proveemos una cota inferior para el primer valor propio del problema de Steklov en un dominio estrellado acotado en $\mathbb{R}^n$. Este resultado extiende a dimensiones altas un estimativo inferior de Kuttler-Sigillito en un dominio estrellado acotado dos dimensional.

Palabras y frases clave. Valor propio, cota inferior, problema de Steklov.

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. The following problem is called the Steklov problem:

$$\begin{aligned} \Delta\varphi &= 0, \qquad \text{in } \ \Omega \\ \frac{\partial\varphi}{\partial\eta} &= \nu, \qquad \varphi \ \text{ on } \ \partial\Omega, \end{aligned} \tag{1}$$

where ν is a real number. This problem was studied by Steklov [7] for bounded domains in the plane. The problem has physical origins; the function φ is a steady state temperature in Ω where the flow over the boundary, $\partial\Omega$, is proportional to the temperature. The set of eigenvalues for the Steklov problem is the same as the set of eigenvalues for the Dirichlet-Neumann function. This function associates to each function u defined on $\partial\Omega$, the normal derivative of its harmonic extension $\widehat{u}$ on Ω. The set of eigenvalues of the Steklov problem consists of an increasing sequence $0 = \nu_0 < \nu_1 < \nu_2 < \cdots$, with $\nu_k \to +\infty$. The first non-zero eigenvalue is known as the first eigenvalue of the Steklov problem; the variational characterization of this eigenvalue is

$$\nu_1 = \min_{\varphi}\left\{ \frac{\int_\Omega \left|\nabla\varphi\right|^2 dv}{\int_{\partial\Omega} \varphi^2 \, d\sigma} : \varphi \in C^\infty\left(\overline{\Omega}\right), \int_{\partial\Omega} \varphi \, d\sigma = 0 \right\}. \tag{2}$$

For the unit ball $B^n \subset \mathbb{R}^n$, the eigenvalues of the Dirichlet-Neumann function are $\nu_k = k$, $k = 0, 1, 2, \ldots$, and the eigenfunctions are given by the space of harmonics homogeneous polynomials of degree k restricted to the boundary of the ball, the $(n-1)$-dimensional unit sphere S^{n-1}. The first Steklov eigenvalue of the n-dimensional ball of radius $r > 0$, B_r, is $\nu_1(B_r) = \frac{1}{r}$ and the coordinate functions $\{x_1, \ldots, x_n\}$ are the respective eigenfunctions.

As in the Dirichlet and the Neumann problem, Steklov geometric estimates have been made in the Steklov problem for the first eigenvalue. For bounded and simply connected domains in the plane xy, in 1954 Weinstock [8] proved that $\nu_1 \leq \frac{2\pi}{L}$, where L represents the perimeter of the boundary curve, with equality if and only if Ω is a disk. In 1970 for convex domains in the plane, Payne [6] proved that $\nu_1 \geq k_o$, where k_o is the minimum value of the curvature on the boundary of the domain. In 1997, Escobar [2] generalized Payne's result to 2-dimensional Riemannian manifolds with non-negative Gaussian curvature and with boundary such that the geodesic curvature k_g is bounded below by a positive constant k_o; with these hypotheses Escobar showed that $\nu_1 \geq k_o$. In higher dimensions, Escobar considered compact manifolds with nonnegative Ricci curvature and again in the spirit of Payne's theorem proved the following result:

Theorem 1.1. *If Ω is an n-dimensional compact Riemannian manifold ($n \geq 3$) with nonnegative Ricci Curvature, with nonempty smooth boundary $\partial\Omega$ and whose second fundamental form π on $\partial\Omega$ satisfies $\pi \geq kI$ for some positive constant k, then*

$$\nu_1 > \frac{k}{2}.$$

For rotationally invariant metrics with nonnegative Ricci curvature in the n-dimensional ball B_r, Montaño [5] proved that $\nu_1 \geq h$ where h is the mean curvature on ∂B_r. For rotationally invariant metrics with nonpositive Ricci curvature in the n-dimensional ball B_r, Montaño [4] proved that $\nu_1 \leq h$ where h is the mean curvature on ∂B_r. In both cases equality holds if and only if (B_r, g) is isometric to the Euclidean ball.

When $\Omega \subset \mathbb{R}^n$ is a star-shaped domain with respect to a point P, which, without loss of generality we can assume that it is the origin, Bramble and

Payne [1] proved that

$$\nu_1 \geq \frac{a^{n-1}}{r_M^{n+1}} h_m,$$

where a is the radius of a ball centered at the origin contained in Ω, r_M is the maximum distance from P to border of Ω, and h_m is the minimum of the function $h : \partial\Omega \to \mathbb{R}$, defined by

$$h(x) = \langle x, \eta \rangle,$$

with η a outer unit normal to $\partial\Omega$.

With a different idea for the 2-dimensional case Kuttler and Sigillito [3] proved that

$$\nu_1 \geq \frac{\left\{1 - \frac{2}{1+\sqrt{1+4\min\left(\frac{R}{R'}\right)^2}}\right\}}{\max\sqrt{R^2 + (R')^2}},$$

where $R(\theta) = |x|$ for any $x \in \partial\Omega$ of the form $x = |x|e^{i\theta}$.

In this paper, following the idea of Sigillito and Kruttler we provide a lower bound for the first Steklov eigenvalue in a star-shaped bounded domain $\Omega \subset \mathbb{R}^n$ with respect to a point P. This improvement of the Kuttler and Sigillito's result depends on two results which relate geometric quantities on $\partial\Omega$ and on $\mathbb{S}^{n-1}$.

2. Preliminaries

Consider on the Euclidean space $(\mathbb{R}^n, \langle\,,\rangle)$ polar coordinates (r, ω), where $r \in \mathbb{R}^+$ and $\omega \in \mathbb{S}^{n-1}$. If $u : U \to \mathbb{S}^{n-1}$ is a local chart of the unit sphere, $\langle\,,\rangle$ writes as

$$\langle\,,\rangle = dr^2 + r^2 \widetilde{g}_{ij} du_i \otimes du_j,$$

where $\widetilde{g}_{ij}$ are the components of the standard round metric on $\mathbb{S}^{n-1}$ in the chart u. Let Ω be a star-shaped bounded domain of $\mathbb{R}^n$ with respect to a point P, which, without loss of generality we assume that it is the origin (see Figure 1). Let $\partial\Omega$ be the boundary of Ω with outer unit normal vector given by η. By the character of star-shaped of Ω there exists a function $R : \mathbb{S}^{n-1} \to \mathbb{R}^+$ such that

$$\Omega = \big\{(r, \omega) \in \mathbb{R}^n \setminus \{0\} : 0 < r < R(\omega)\big\}$$

and

$$\partial\Omega = \big\{(r, \omega) \in \mathbb{R}^n \setminus \{0\} : r = R(\omega)\big\}.$$

In this work we assume that the boundary of Ω is smooth. In the following proposition we get a result that relates the function R and its gradient, $\overline{\nabla} R$, on the boundary of Ω.

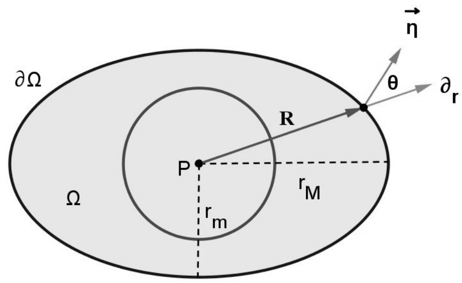

Figure 1. Star-shaped domain Ω.

Proposition 2.1. *Let $G = (g_{ij})$ and $\widetilde{G} = \big(\widetilde{g}_{ij}\big)$ be respectively the matrices of the first fundamental forms for $\partial\Omega$ and S^{n-1}, in the given coordinate systems. Then*

$$\frac{\left|\overline{\nabla} R\right|^2}{R^2} = \tan^2(\theta), \tag{3}$$

where θ is the angle between the outer unit normal vector η and the radial vector field ∂_r.

Proof. The map $\Psi : \mathbb{S}^{n-1} \to \mathbb{R}^n$, given by $\Psi(\omega) = \big(R(\omega), \omega\big)$ realizes the embedding of $\partial\Omega$. The induced metric on $\partial\Omega$ therefore write, in the chart u as

$$\begin{aligned}\Psi^*\langle\,,\rangle &= \Psi^*\big(dr^2 + r^2\widetilde{g}_{ij}du_i \otimes du_j\big) = dR^2 + R^2\widetilde{g}_{ij}du_i \otimes du_j \\ &= \big(R_iR_j + R^2\widetilde{g}_{ij}\big)du_i \otimes du_j = g_{ij}du_i \otimes du_j.\end{aligned}$$

Note that, setting $t = \log R$, we can write

$$g_{ij} = e^{2t}\big(t_it_j + \widetilde{g}_{ij}\big). \tag{4}$$

The inverse metric g^{ij} therefore writes as $g^{ij} = e^{-2t}\Big(\widetilde{g}^{ij} - \frac{t^it^j}{W^2}\Big)$, where $W = \sqrt{1 + \left|\overline{\nabla} t\right|^2} = \frac{\sqrt{r^2+|\nabla r|^2}}{r}$, $t^j = \widetilde{g}^{jk}t_k$ and $\overline{\nabla}$ is the connection on $\big(\mathbb{S}^{n-1}, \widetilde{g}_{ij}\big)$.

Since $\partial\Omega$ is the zero set of the function $F(r,\omega) = r - R(\omega)$, then the normal vector η is the normalized gradient ∇F, hence

$$\eta = \frac{1}{W}(\partial r - \frac{R^j}{R^2}\partial_j).$$

It therefore follows that

$$\cos(\theta) = \langle \eta, \partial_r\rangle = \left\langle \frac{1}{W}(\partial r - \frac{R^j}{R^2}\partial_j), \partial_r \right\rangle = \frac{1}{W}.$$

Solving $W^2 \cos^2\theta = 1$ in terms of $\left|\overline{\nabla} t\right|^2$ we deduce that

$$\frac{\left|\overline{\nabla} R\right|^2}{R^2} = \left|\overline{\nabla} t\right|^2 = \frac{1-\cos^2\theta}{\cos^2\theta} = \tan^2\theta,$$

as claimed. ☑

It is natural to ask for the relationship between the elements of area $\sqrt{g} = \det(G)$ and $\sqrt{\widetilde{g}} = \det(\widetilde{G})$. In this direction we have the following comparison theorem; although its proof is standard, we include it for the convenience of the reader.

Proposition 2.2.

$$\frac{\sqrt{g}}{\sqrt{\widetilde{g}}} = R^{n-1}\sqrt{\frac{\left|\overline{\nabla} R\right|^2}{R^2} + 1}. \tag{5}$$

Proof. Let us take a local chart of the unit sphere $u : U \to \mathbb{S}^{n-1}$ such that the matrix $\widetilde{G} = \left(\widetilde{g}_{ij}\right)$ of the first fundamental form in the given chart is diagonal.

Consider the $(n-1)\times(n-1)$ matrix $B = (b_{ij})$, where $b_{ij} = \sqrt{\widetilde{g}_{ij}}$ and the vector $t = \left(t_1, t_2, \dots, t_{n-1}\right)^t$.

By formula (4) we have

$$g_{ij} = e^{2t}\left(AA^t\right)_{ij},$$

where A is the $(n-1)\times n$ matrix whose entries are

$$A = [B \mid t].$$

The determinant $\det g$ can be therefore obtained easily via Binet theorem for $\det\left(AA^t\right)$

$$\begin{aligned}
\det(G) &= e^{2(n-1)t}\left(\det\widetilde{G} + \sum_{i=1}^{n-1}\frac{t_i^2 \det\widetilde{G}}{\widetilde{g}_{ii}}\right)\\
&= e^{2(n-1)t}\det\widetilde{G}\left(1 + \sum_{i=1}^{n-1} t_i^2 \widetilde{g}^{ii}\right)\\
&= e^{2(n-1)t}\det\widetilde{G}\left(1 + \sum_{i=1}^{n-1} t_i t^i\right)\\
&= e^{2(n-1)t}\det\widetilde{G}\left(1 + \left|\overline{\nabla} t\right|^2\right)
\end{aligned}$$

reaching the geometric relationship

$$\frac{\sqrt{g}}{\sqrt{\widetilde{g}}} = R^{n-1}\sqrt{\frac{\left|\overline{\nabla}R\right|^2}{R^2}+1}. \tag{6}$$

☑

3. Main Result

Using the results of the previous section, we establish the following theorem, where we find a lower bound for the first eigenvalue of the Steklov problem in a star-shaped bounded domain.

Theorem 3.1. *Let Ω be a bounded star-shaped domain of R^n with smooth boundary $\partial\Omega$ and outer unit normal η. If $0 \le \theta \le \alpha < \frac{\pi}{2}$, where $\cos(\theta) = \langle \eta, \partial r\rangle$, then the first eigenvalue of Steklov for Ω, $\nu_1(\Omega)$, satisfies the inequality*

$$\nu_1(\Omega) \ge \left(\frac{r_m^{n-2}}{r_M^{n-1}}\right)\frac{\left\{2+a-\sqrt{a^2+4a}\right\}}{2\sqrt{a+1}}, \tag{7}$$

where $a = \tan^2(\alpha)$, $r_m := \min\limits_U R$ and $r_M := \max\limits_U R$.

Proof. Without loss of generality let us assume that Ω is a star-shaped domain with respect to the origin, $\xi : U \to S^{n-1}$ is a standard parametrization of $S^{n-1} \subseteq \mathbb{R}^n$ and $y : U \to \partial\Omega$ is the associated parametrization to the boundary of Ω. If we define $R : U \to \mathbb{R}$ for $R(u) = |y(u)|$, then $y = R\xi$. Considering the spherical coordinates $x = r\xi(u)$, $\Omega = \big\{(u,r) : u \in U, 0 < r < R(u)\big\}$ and $dx = r^{n-1}\sqrt{\widetilde{g}}\,dr\,du$. For $\varphi : \Omega \to R$, we have the Rayleigh's quotient

$$R[\varphi] = \frac{\int\limits_{\Omega} |\nabla\varphi|^2\,dx}{\int\limits_{\partial\Omega} \varphi^2\,d\sigma_x} = \frac{\int\limits_0^{R(u)}\int\limits_U \left\{\left(\frac{\partial\varphi}{\partial r}\right)^2 + \frac{1}{r^2}\left|\overline{\nabla}\varphi\right|^2\right\} r^{n-1}\sqrt{\widetilde{g}}\,du\,dr}{\int\limits_U \varphi^2\sqrt{g}\,du},$$

where $g_{ij} = \frac{\partial R}{\partial u_i}\frac{\partial R}{\partial u_j} + R^2\widetilde{g}_{ij}$.

Making the change of variables $u = u$ and $r = \rho R(u)$ we obtain

$$R[\varphi] = \left(\int\limits_U \varphi^2\sqrt{g}\,du\right)^{-1}\int\limits_0^1\int\limits_U \left(\left(\frac{\partial\varphi}{\partial\rho}\right)^2 + \frac{1}{\rho^2}\left|\overline{\nabla}\varphi\right|^2 + \right.$$
$$\left.\frac{1}{\rho^2}\left\{-\frac{2\rho}{R}\frac{\partial\varphi}{\partial\rho}\langle\overline{\nabla}\varphi,\overline{\nabla}R\rangle + \frac{\rho^2}{R^2}\left(\frac{\partial\varphi}{\partial\rho}\right)^2\left|\overline{\nabla}R\right|^2\right\}\right)\rho^{n-1}R^{n-2}\sqrt{\widetilde{g}}\,du\,d\rho.$$

From the Cauchy-Schwarz inequality,

$$-\frac{2\rho}{R}\frac{\partial\varphi}{\partial\rho}\left\langle\overline{\nabla}\varphi,\overline{\nabla}R\right\rangle\geq-\left(\gamma^2\left|\overline{\nabla}\varphi\right|^2+\frac{\rho^2}{\gamma^2R^2}\left(\frac{\partial\varphi}{\partial\rho}\right)^2\left|\overline{\nabla}R\right|^2\right),$$

for any function γ^2. From here,

$$R[\varphi]\geq \frac{\int_0^1\int_U R^{n-2}\left\{\left(1-\frac{1-\gamma^2}{\gamma^2}\frac{\left|\overline{\nabla}R\right|^2}{R^2}\right)\left(\frac{\partial\varphi}{\partial\rho}\right)^2+\frac{1}{\rho^2}\left(1-\gamma^2\right)\left|\overline{\nabla}\varphi\right|^2\right\}\rho^{n-1}\sqrt{\widetilde{g}}\,du\,d\rho}{\int_U\varphi^2\sqrt{g}\,du}.$$

Making $\frac{1-\gamma^2}{\gamma^2}=\beta^2$, it follows that

$$R[\varphi]\geq\frac{\int_0^1\int_U R^{n-2}\left\{\left(1-\beta^2\frac{\left|\overline{\nabla}R\right|^2}{R^2}\right)\left(\frac{\partial\varphi}{\partial\rho}\right)^2+\frac{1}{\rho^2}\frac{\beta^2}{1+\beta^2}\left|\overline{\nabla}\varphi\right|^2\right\}\rho^{n-1}\sqrt{\widetilde{g}}\,du\,d\rho}{\int_U\varphi^2\sqrt{g}\,du}.$$

From the equality (3), taking $a=\tan^2(\alpha)$, we arrive to

$$\begin{aligned}
R[\varphi]&\geq\frac{\int_0^1\int_U R^{n-2}\left\{\left(1-\beta^2a\right)\left(\frac{\partial\varphi}{\partial\rho}\right)^2+\frac{1}{\rho^2}\frac{\beta^2}{1+\beta^2}\left|\overline{\nabla}\varphi\right|^2\right\}\rho^{n-1}\sqrt{\widetilde{g}}\,du\,d\rho}{\int_U\varphi^2\sqrt{g}\,du}\\
&=\frac{\int_0^1\int_U R^{n-2}\left\{\left(1-\beta^2a\right)\left(\frac{\partial\varphi}{\partial\rho}\right)^2+\frac{1}{\rho^2}\frac{\beta^2}{1+\beta^2}\left|\overline{\nabla}\varphi\right|^2\right\}\rho^{n-1}\sqrt{\widetilde{g}}\,du\,d\rho}{\int_U\varphi^2\left(\frac{\sqrt{g}}{\sqrt{\widetilde{g}}}\right)\sqrt{\widetilde{g}}\,du}\\
&=\frac{\int_0^1\int_U R^{n-2}\left\{\left(1-\beta^2a\right)\left(\frac{\partial\varphi}{\partial\rho}\right)^2+\frac{1}{\rho^2}\frac{\beta^2}{1+\beta^2}\left|\overline{\nabla}\varphi\right|^2\right\}\rho^{n-1}\sqrt{\widetilde{g}}\,du\,d\rho}{\int_U\varphi^2\left(\sqrt{\frac{\left|\overline{\nabla}R\right|^2}{R^2}+1}\right)R^{n-1}\sqrt{\widetilde{g}}\,du}\\
&\geq\frac{\int_0^1\int_U R^{n-2}\left\{\left(1-\beta^2a\right)\left(\frac{\partial\varphi}{\partial\rho}\right)^2+\frac{1}{\rho^2}\frac{\beta^2}{1+\beta^2}\left|\overline{\nabla}\varphi\right|^2\right\}\rho^{n-1}\sqrt{\widetilde{g}}\,du\,d\rho}{\int_U\varphi^2\left(\sqrt{a+1}\right)R^{n-1}\sqrt{\widetilde{g}}\,du}.
\end{aligned}$$

Since $r_m:=\min\limits_U R$ and $r_M:=\max\limits_U R$, we have

$$R[\varphi] \geq \frac{r_m^{n-2}}{r_M^{n-1}\sqrt{a+1}} \frac{\int\limits_0^1 \int\limits_U \left\{\left(1-\beta^2 a\right)\left(\frac{\partial\varphi}{\partial\rho}\right)^2 + \frac{1}{\rho^2}\frac{\beta^2}{1+\beta^2}\left|\overline{\nabla}\varphi\right|^2\right\}\rho^{n-1}\sqrt{\widetilde{g}}\,du\,d\rho}{\int\limits_U \varphi^2\sqrt{\widetilde{g}}\,du}.$$

Solving for β^2 the equation $1-\beta^2 a = \frac{\beta^2}{1+\beta^2}$ we obtain

$$1-\beta^2 a = \frac{\beta^2}{1+\beta^2} = \frac{2+a-\sqrt{a^2+4a}}{2} > 0.$$

Consequently,

$$R[\varphi] \geq \left(\frac{r_m^{n-2}}{r_M^{n-1}}\right)\frac{\left\{2+a-\sqrt{a^2+4a}\right\}}{2\sqrt{a+1}} \frac{\int\limits_0^1 \int\limits_U \left\{\left(\frac{\partial\varphi}{\partial\rho}\right)^2 + \frac{1}{\rho^2}\left|\overline{\nabla}\varphi\right|^2\right\}\rho^{n-1}\sqrt{\widetilde{g}}\,du\,d\rho}{\int\limits_U \varphi^2\sqrt{\widetilde{g}}\,du}$$

$$= \left(\frac{r_m^{n-2}}{r_M^{n-1}}\right)\frac{\left\{2+a-\sqrt{a^2+4a}\right\}}{2\sqrt{a+1}} \frac{\int\limits_{B_1(0)} \left|\nabla\varphi\right|^2 dx}{\int\limits_{S^{n-1}} \varphi^2\, d\sigma_x}.$$

If φ is feasible for the ball $B_1(0)$ then

$$R[\varphi] \geq \left(\frac{r_m^{n-2}}{r_M^{n-1}}\right)\frac{\left\{2+a-\sqrt{a^2+4a}\right\}}{2\sqrt{a+1}}.$$

If we take $\varphi = \varphi_1 - \frac{\int\limits_{S^{n-1}} \varphi_1\, d\sigma_x}{\int\limits_{S^{n-1}} d\sigma_x}$, where φ_1 is a first eigenfunction for the Steklov problem on Ω we get

$$R[\varphi] = R[\varphi_1 - \overline{\varphi_1}] \geq \left(\frac{r_m^{n-2}}{r_M^{n-1}}\right)\frac{\left\{2+a-\sqrt{a^2+4a}\right\}}{2\sqrt{a+1}} \frac{\int_{B_1}\left|\nabla\varphi\right|^2 dx}{\int\limits_{S^{n-1}} \varphi^2\, d\sigma_x}$$

$$\frac{\int\limits_\Omega \left|\nabla\varphi_1\right|^2 dx}{\int_{\partial\Omega}(\varphi_1-\overline{\varphi_1})^2\, d\sigma_x} \geq \left(\frac{r_m^{n-2}}{r_M^{n-1}}\right)\frac{\left\{2+a-\sqrt{a^2+4a}\right\}}{2\sqrt{a+1}}$$

$$\frac{\int\limits_\Omega \left|\nabla\varphi_1\right|^2 dx}{\int\limits_{\partial\Omega}\left(\varphi_1^2+\overline{\varphi_1}^2\right) d\sigma_x} \geq \left(\frac{r_m^{n-2}}{r_M^{n-1}}\right)\frac{\left\{2+a-\sqrt{a^2+4a}\right\}}{2\sqrt{a+1}},$$

hence

$$\frac{\int\limits_{\Omega}\left|\nabla\varphi_1\right|^2 dx}{\int\limits_{\partial\Omega}\varphi_1^2\,d\sigma_x} \geq \frac{\int\limits_{\Omega}\left|\nabla\varphi_1\right|^2 dx}{\int\limits_{\partial\Omega}\left(\varphi_1^2+\overline{\varphi_1}^2\right)d\sigma_x}$$
$$\geq \left(\frac{r_m^{n-2}}{r_M^{n-1}}\right)\frac{\left\{2+a-\sqrt{a^2+4a}\right\}}{2\sqrt{a+1}},$$

and therefore

$$\nu_1(\Omega) \geq \left(\frac{r_m^{n-2}}{r_M^{n-1}}\right)\frac{\left\{2+a-\sqrt{a^2+4a}\right\}}{2\sqrt{a+1}},$$

where $a=\tan^2(\alpha)$. ☑

Observe from here that, in the ball of radius r, $a=0$, and in this case our estimative is sharp.

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