

Sectional-Anosov Flows in Higher Dimensions

Flujos seccionales Anosov en dimensiones superiores

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ABSTRACT. A *sectional-Anosov flow* on a manifold is a C^1 vector field inwardly transverse to the boundary for which the maximal invariant is sectional hyperbolic [10]. We prove that every attractor of every vector field C^1 close to a transitive sectional-Anosov flow with singularities on a compact manifold has a singularity. This extends the three-dimensional result obtained in [9].

Key words and phrases. Transitive, Maximal invariant, Sectional-Anosov flow.

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RESUMEN. Un *flujo seccional-Anosov* sobre una variedad es un C^1 campo vectorial transversal a la frontera apuntando hacia el interior, para el cual su conjunto maximal invariante es un conjunto seccional hiperbólico [10]. Probamos que todo atractor de todo campo vectorial C^1 próximo a un flujo seccional-Anosov transitivo con singularidades sobre una variedad compacta tiene una singularidad. Este resultado extiende el resultado tres-dimensional obtenido en [9].

Palabras y frases clave. Transitivo, maximal invariante, flujo seccional-Anosov.

1. Introduction

The *sectional-Anosov flows* were introduced in [10] as a generalization of the *Anosov flows*. These also includes the *saddle-type hyperbolic attracting sets* and the *geometric and multidimensional Lorenz attractors* [1, 5, 7]. Some properties of these flows have been shown in the literature [2, 4]. For instance, [9] proved that every attractor of every vector field C^1 close to a transitive sectional-Anosov flow with singularities on a compact 3-manifold has a singularity. Moreover, [3] generalized this result from transitive to nonwandering

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ones. In this paper we further extend [9] but now to higher dimensions. More precisely, we prove that every attractor of every vector field C^1 close to a transitive sectional-Anosov flow with singularities of a compact manifold has a singularity. Let us state our result in a precise way.

Consider a compact Riemannian manifold M of dimension $n \geq 3$ (a *compact n -manifold* for short). We denote by ∂M the boundary of M . Let $\mathcal{X}^1(M)$ be the space of C^1 vector fields in M endowed with the C^1 topology. Fix $X \in \mathcal{X}^1(M)$, inwardly transverse to the boundary ∂M and denote by X_t the flow of X , $t \in \mathbb{R}$.

The ω -limit set of $p \in M$ is the set $\omega_X(p)$ formed by those $q \in M$ such that $q = \lim_{n \rightarrow \infty} X_{t_n}(p)$ for some sequence $t_n \rightarrow \infty$.

Given $\Lambda \in M$ compact, we say that Λ is *invariant* if $X_t(\Lambda) = \Lambda$ for all $t \in \mathbb{R}$. We also say that Λ is *transitive* if $\Lambda = \omega_X(p)$ for some $p \in \Lambda$; *singular* if it contains a singularity and *attracting* if $\Lambda = \bigcap_{t>0} X_t(U)$ for some compact neighborhood U of it. This neighborhood is often called *isolating block*. It is well known that the isolating block U can be chosen to be positively invariant, i.e., $X_t(U) \subset U$ for all $t > 0$. An *attractor* is a transitive attracting set. An attractor is *nontrivial* if it is not a closed orbit.

Note that the set formed by a single singularity is a transitive set and in this case such transitive set is trivial.

The *maximal invariant* set of X is defined by $M(X) = \bigcap_{t \geq 0} X_t(M)$.

We denote by $m(L)$ the minimum norm of a linear operator L , i.e., $m(L) = \inf_{v \neq 0} \frac{\|Lv\|}{\|v\|}$.

Definition 1.1. A compact invariant set Λ is *partially hyperbolic* if there is a continuous invariant splitting

$$T_\Lambda M = E^s \oplus E^c$$

such that the following properties hold for some positive constants C, λ :

- (1) E^s is *contracting*, i.e.,

$$\|DX_t(x)|_{E_x^s}\| \leq Ce^{-\lambda t},$$

for all $x \in \Lambda$ and $t > 0$.

- (2) E^s *dominates* E^c , i.e.,

$$\frac{\|DX_t(x)|_{E_x^s}\|}{m(DX_t(x)|_{E_x^c})} \leq Ce^{-\lambda t},$$

for all $x \in \Lambda$ and $t > 0$.

We say that the central subbundle E_x^c of Λ as above is *sectionally expanding* if for all two-dimensional subspace L_x of E_x^c

$$\dim(E_x^c) \geq 2 \quad \text{and} \quad |J(DX_t(x)|_{L_x})| \geq C^{-1}e^{\lambda t}, \quad \forall x \in \Lambda \quad \text{and} \quad t > 0,$$

where $J(\cdot)$ is the jacobian.

Definition 1.2. A *sectional hyperbolic set* is a partially hyperbolic set whose singularities are hyperbolic and whose central subbundle is sectionally-expanding.

Recall that a singularity of a vector field is hyperbolic if the eigenvalues of its linear part have non zero real part.

Definition 1.3. We say that X is a *sectional-Anosov flow* if $M(X)$ is a sectional hyperbolic set.

Our result is the following.

Theorem 1.4. *Let X be a transitive sectional-Anosov flow with singularities of a compact n -manifold. Then, every attractor of every vector field C^1 close to X has a singularity.*

The proof of this theorem follows closely that of [9]. More precisely, we assume by contradiction that there exists a sequence X^n of vector fields C^1 close to X each one exhibiting a non-singular attractor A^n . We then prove that A^n accumulates on a singularity of X and, consequently, for n large, we will prove that the corresponding attractor A^n does contain a singularity. This give us the desired contradiction. To prove such assertions we will extend some tools in [9] including the definitions of both Lorenz-like singularity and singular cross-section.

2. Lorenz-Like Singularities and Singular Cross-Sections in Higher Dimension

Let M be a compact Riemannian n -manifold, $n \geq 3$. Fix $X \in \mathcal{X}^1(M)$ inwardly transverse to ∂M . Denote by X_t the flow of X , $t \in \mathbb{R}$, and by $M(X)$ the maximal invariant of X .

Definition 2.1. A compact invariant set Λ of X is *hyperbolic* if there are a continuous tangent bundle invariant decomposition $T_\Lambda M = E^s \oplus E^X \oplus E^u$ and positive constants C, λ such that

- E^X is the vector field direction over Λ .
- E^s is *contracting*, i.e., $\|DX_t(x)|_{E_x^s}\| \leq Ce^{-\lambda t}$, for all $x \in \Lambda$ and $t > 0$.
- E^u is *expanding*, i.e., $\|DX_{-t}(x)|_{E_x^u}\| \leq Ce^{-\lambda t}$, for all $x \in \Lambda$ and $t > 0$.

A closed orbit or attractor is hyperbolic if it does as a compact invariant set.

It follows from the stable manifold theory [8] that if p belongs to a hyperbolic set Λ , then the sets

$$\begin{aligned} W_X^{ss}(p) &= \{x : d(X_t(x), X_t(p)) \rightarrow 0, t \rightarrow \infty\} \quad \text{and} \\ W_X^{uu}(p) &= \{x : d(X_t(x), X_t(p)) \rightarrow 0, t \rightarrow -\infty\} \end{aligned}$$

are C^1 immersed submanifolds of M which are tangent at p to the subspaces E_p^s and E_p^u of T_pM respectively. Similarly, the sets

$$\begin{aligned} W_X^s(p) &= \bigcup_{t \in \mathbb{R}} W_X^{ss}(X_t(p)) \quad \text{and} \\ W_X^u(p) &= \bigcup_{t \in \mathbb{R}} W_X^{uu}(X_t(p)) \end{aligned}$$

are also C^1 immersed submanifolds tangent to $E_p^s \oplus E_p^X$ and $E_p^X \oplus E_p^u$ at p respectively. Moreover, for every $\epsilon > 0$ we have that

$$\begin{aligned} W_X^{ss}(p, \epsilon) &= \{x : d(X_t(x), X_t(p)) \leq \epsilon, \forall t \geq 0\} \quad \text{and} \\ W_X^{uu}(p, \epsilon) &= \{x : d(X_t(x), X_t(p)) \leq \epsilon, \forall t \leq 0\} \end{aligned}$$

are closed neighborhoods of p in $W_X^{ss}(p)$ and $W_X^{uu}(p)$ respectively.

There is also a stable manifold theorem in the case when X is sectional-Anosov. Indeed, denoting by $T_{M(X)}M = E_{M(X)}^s \oplus E_{M(X)}^c$ the corresponding sectional-hyperbolic splitting over $M(X)$ we have from [8] that the contracting subbundle $E_{M(X)}^s$ can be extended to a contracting subbundle E_U^s in M , where U is a neighborhood of $M(X)$. Moreover, such an extension is tangent to a continuous foliation denoted by W^{ss} (or W_X^{ss} to indicate dependence on X). By adding the flow direction to W^{ss} we obtain a continuous foliation W^s (or W_X^s) now tangent to $E_M^s \oplus E_M^X$. Unlike the Anosov case W^s may have singularities, all of which being the leaves $W^{ss}(\sigma)$ passing through the singularities σ of X . Note that W^s is transverse to ∂M because it contains the flow direction (which is transverse to ∂M by definition).

It turns out that every singularity σ of a sectional-Anosov flow X satisfies $W_X^{ss}(\sigma) \subset W_X^s(\sigma)$. Furthermore, there are two possibilities for such a singularity, namely, either $\dim(W_X^{ss}(\sigma)) = \dim(W_X^s(\sigma))$ (and so $W_X^{ss}(\sigma) = W_X^s(\sigma)$) or $\dim(W_X^s(\sigma)) = \dim(W_X^{ss}(\sigma)) + 1$. In the later case we call it Lorenz-like according to the following definition.

Definition 2.2. We say that a singularity σ of a sectional-Anosov flow X is *Lorenz-like* if $\dim(W^s(\sigma)) = \dim(W^{ss}(\sigma)) + 1$.

Hereafter, we will denote $\dim(W_X^{ss}(\sigma)) = s$, $\dim(W_X^u(\sigma)) = u$ and therefore $\dim(W_X^s(\sigma)) = s + 1$ by definition. Moreover $W_X^{ss}(\sigma)$ separates $W_{loc}^s(\sigma)$ in two connected components denoted by $W_{loc}^{s,t}(\sigma)$ and $W_{loc}^{s,b}(\sigma)$ respectively.

Next we define singular cross-section in the higher dimensional context. First, we will denote a cross-section by Σ and its boundary by $\partial\Sigma$. Also, the hypercube $I^k = [-1, 1]^k$ will be submanifold of dimension k , with $k \in \mathbb{N}$.

Thus, we begin by considering $B^u[0, 1] \approx I^u$ and $B^{ss}[0, 1] \approx I^s$ where

$B^{ss}[0, 1]$ is the ball centered at zero and radius 1 contained in $\mathbb{R}^{\dim(W^{ss}(\sigma))} = \mathbb{R}^s$ and

$B^u[0, 1]$ is the ball centered at zero and radius 1 contained in $\mathbb{R}^{\dim(W^u(\sigma))} = \mathbb{R}^{n-s-1} = \mathbb{R}^u$.

Definition 2.3. A *singular cross-section* of a Lorenz-like singularity σ consists of a pair of submanifolds Σ^t, Σ^b , where Σ^t, Σ^b are cross-sections and

Σ^t is transversal to $W_{loc}^{s,t}(\sigma)$.

Σ^b is transversal to $W_{loc}^{s,b}(\sigma)$.

Note that every singular cross-section contains a pair singular submanifolds l^t, l^b defined as the intersection of the local stable manifold of σ with Σ^t, Σ^b respectively and additionally $\dim(l^*) = \dim(W^{ss}(\sigma))$ ($* = t, b$).

Thus, a singular cross-section Σ^* will be a *hypercube of dimension* $(n - 1)$, i.e., diffeomorphic to $B^u[0, 1] \times B^{ss}[0, 1]$. Let $f : B^u[0, 1] \times B^{ss}[0, 1] \rightarrow \Sigma^*$ be the diffeomorphism, such that

$$f(\{0\} \times B^{ss}[0, 1]) = l^*$$

and $\{0\} = 0 \in \mathbb{R}^u$. Define

$$\partial\Sigma^* = \partial^h\Sigma^* \cup \partial^v\Sigma^*$$

by

$$\partial^h\Sigma^* = \{\text{union of the boundary submanifolds which are transverse to } l^*\}$$

and

$$\partial^v\Sigma^* = \{\text{union of the boundary submanifolds which are parallel to } l^*\}.$$

From this decomposition we obtain that

$$\begin{aligned} \partial^h\Sigma^* = & \\ & \left(I^u \times \left[\bigcup_{j=0}^{s-1} (I^j \times \{-1\} \times I^{s-j-1}) \right] \right) \cup \left(I^u \times \left[\bigcup_{j=0}^{s-1} (I^j \times \{1\} \times I^{s-j-1}) \right] \right) \end{aligned}$$

and

$$\partial^v \Sigma^* = \left(\left[\bigcup_{j=0}^{u-1} (I^j \times \{-1\} \times I^{u-j-1}) \right] \times I^s \right) \cup \left(\left[\bigcup_{j=0}^{u-1} (I^j \times \{1\} \times I^{u-j-1}) \right] \times I^s \right),$$

where $I^0 \times I = I$.

Hereafter we denote $\Sigma^* = B^u[0, 1] \times B^{ss}[0, 1]$.

3. Sectional Hyperbolic Sets in Higher Dimension

In this section we prove a preliminary result for transitive sectional-Anosov flows. To begin with, we present the following useful property.

Lemma 3.1. *Let X be a sectional-Anosov flow, X a C^1 vector field in M . If Y is C^1 close to X , then every nonempty, compact, non singular, invariant set H of Y is hyperbolic saddle-type (i.e. $E^s \neq 0$ and $E^u \neq 0$).*

Proof. See [11]. The proof in [11] is made in dimension three, but the same proof yields the same conclusion in any dimension. \checkmark

Lemma 3.2. *Let X be a transitive sectional-Anosov flow C^1 in M . If $O \subset M(X)$ is a periodic orbit of X , then O is a hyperbolic saddle-type periodic orbit. In addition, if $p \in O$ then the set*

$$\{q \in W_X^{uu}(p) : M(X) = \omega_X(q)\}$$

is dense in $W_X^{uu}(p)$.

Proof. By Lemma 3.1, we have that O is hyperbolic and saddle-type. Let W be an open set in $W_X^{uu}(p)$. This set W exists since the point p belongs to the periodic orbit O which is hyperbolic. Define

$$B = \bigcup_{0 \leq t \leq 1} X_t(W).$$

This set has dimension at least two, and so,

$$B' = \bigcup_{x \in B} W_X^{ss}(x)$$

contains an open set V with $B \cap V \neq \emptyset$.

Since $M(X)$ is the maximal invariant of X , $B \subset W_X^u(p)$ and $p \in O$, we obtain $B \cap V \subset M(X)$. Let $q \in M(X)$ such that $M(X) = \omega_X(q)$. Then, the forward orbit of q intersects V and so it intersects B' too. It follows from the definition of B' that the positive orbit of q is asymptotic to the forward orbit of some $q' \in B$. In particular, $M(X) = \omega_X(q) = \omega_X(q')$. This proves that $\{q \in W_X^{uu}(p) : M(X) = \omega_X(q)\}$ is dense in $M(X)$ as desired. \checkmark

The following theorem appears in [4].

Theorem 3.3. *Let X be a transitive sectional-Anosov flow C^1 for M . Then, every $\sigma \in \text{Sing}(X) \cap M(X)$ is Lorenz-like and satisfies*

$$M(X) \cap W_X^{ss}(\sigma) = \{\sigma\}.$$

Proof. We begin by proving two claims.

Claim 3.4. If $x \in (M(X) \setminus \text{Sing}(X))$, then $X(x) \notin E_x^s$.

Proof. Suppose by contradiction that there is $x_0 \in (M(X) \setminus \text{Sing}(X))$ such that $X(x_0) \in E_{x_0}^s$. Then, $X(x) \in E_x^s$ for every x in the orbit of x_0 since $E_{M(X)}^s$ is invariant. So, by continuity,

$$X(x) \in E_x^s \text{ for every } x \in \alpha(x_0). \quad (1)$$

It follows that $\omega(x)$ is a singularity for all $x \in \alpha(x_0)$. In particular, $\alpha(x_0)$ contains a singularity σ which is necessary hyperbolic of saddle-type.

Now we have two cases: $\alpha(x_0) = \{\sigma\}$ or not.

If $\alpha(x_0) = \{\sigma\}$ then $x_0 \in W^u(\sigma)$. For all $t \in \mathbb{R}$ define the unitary vector

$$v^t = \frac{DX_t(x_0)(X(x_0))}{\|DX_t(x_0)(X(x_0))\|}.$$

It follows that

$$v^t \in T_{X_t(x_0)}W^u(\sigma) \cap E_{X_t(x_0)}^s, \quad \forall t \in \mathbb{R}.$$

Take a sequence $t_n \rightarrow \infty$ such that the sequence v^{-t_n} converges to v^∞ . Clearly v^∞ is an unitary vector and, since $X_{-t_n}(x_0) \rightarrow \sigma$ and E^s is continuous we obtain

$$v^\infty \in T_\sigma W^u(\sigma) \cap E_\sigma^s.$$

Therefore v^∞ is an unitary vector which is simultaneously expanded and contracted by $DX_t(\sigma)$, a contradiction. This contradiction shows the result in the first case.

For the second case, we assume $\alpha(x_0) \neq \{\sigma\}$. Then, $(W^u(\sigma) \setminus \{\sigma\}) \cap \alpha(x_0) \neq \emptyset$. Pick $x_1 \in (W^u(\sigma) \setminus \{\sigma\}) \cap \alpha(x_0)$. It follows from (1) that $X(x_1) \in E_{x_1}^s$ and then we get a contradiction as in the first case replacing x_0 by x_1 . This contradiction proves the claim. \square

Claim 3.5. If $\sigma \in \text{Sing}(X)$, then $M(X) \cap W^{ss}(\sigma) = \{\sigma\}$.

Proof. Take $x \in W^{ss}(\sigma) \setminus \{\sigma\}$. Then, $E_x^s = T_x W^{ss}(\sigma)$. Moreover, since $W^{ss}(\sigma)$ is an invariant, we obtain $X(x) \in T_x W^{ss}(\sigma)$. We conclude that $X(x) \in E_x^s$ for all $x \in W^{ss}(\sigma)$ and now Claim 3.4 applies. \checkmark

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This theorem implies the following two useful properties.

Proposition 3.6. *Let X be a transitive sectional-Anosov flow C^1 of M . Let σ be a singularity of X in $M(X)$ (so σ is Lorenz-like by Theorem 3.3). Then, there is a singular-cross section Σ^t, Σ^b of σ in M such that.*

$$(M(Y)) \cap (\partial^h \Sigma^t \cup \partial^h \Sigma^b) = \emptyset,$$

for every C^r vector field Y close to X .

Proof. See [9]. \checkmark

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Let σ be a Lorenz-like singularity of a C^1 vector field X in $\mathcal{X}^1(M)$, and Σ^t, Σ^b be a singular cross-section of σ . Thus for σ we recall that,

$$\begin{aligned} \dim(W_X^{ss}(\sigma)) &= s, \quad \text{then} \\ \dim(W_X^s(\sigma)) &= s+1, \quad \text{and} \quad \dim(W_X^u(\sigma)) = n-s-1, \\ \dim(\Sigma^*) &= s+(n-s-1) = n-1. \end{aligned} \quad (2)$$

Since $\Sigma^* = B^u[0,1] \times B^{ss}[0,1]$, we will set up a family of singular cross-sections as follows: Given $0 < \Delta \leq 1$ small, we define $\Sigma^{*,\Delta} = B^u[0,\Delta] \times B^{ss}[0,1]$, such that

$$l^* \subset \Sigma^{*,\Delta} \subset \Sigma^*, \quad \text{i.e.}$$

$$\begin{aligned} (l^* = \{0\} \times B^{ss}[0,1]) \subset (\Sigma^{*,\Delta} = B^u[0,\Delta] \times B^{ss}[0,1]) \subset \\ (\Sigma^* = B^u[0,1] \times B^{ss}[0,1]), \end{aligned}$$

where we fix a coordinate system (x^*, y^*) in Σ^* ($* = t, b$). We will assume that $\Sigma^* = \Sigma^{*,1}$.

We will use this notation throughout of next lemma and the Theorem 1.4 proof.

Lemma 3.7. *Let X be a transitive sectional-Anosov flow C^1 of M . Let σ be a singularity of X in $M(X)$. Let Y^n be a sequence of vector fields converging to X in the C^1 topology. Let O_n be a periodic orbit of Y^n such that the sequence $\{O_n : n \in \mathbb{N}\}$ accumulates on σ . If $0 < \Delta \leq 1$ and Σ^t, Σ^b is a singular cross-section of σ , then there is n such that either*

$$O_n \cap \text{int}(\Sigma^{t,\Delta}) \neq \emptyset$$

or

$$O_n \cap \text{int}(\Sigma^{b,\Delta}) \neq \emptyset.$$

Proof. Since O_n accumulates on $\sigma \in M(X)$ and $M(X)$ is maximal invariant, we have that $O_n \subset M(X)$ for all n large (recall $Y^n \rightarrow X$ as $n \rightarrow \infty$). Let us fix a fundamental domain D_ϵ of the vector field's flow X_t restricted to the local stable manifold $W_{loc}^s(\sigma)$ ([12]) for $\epsilon > 0$ as follows (See Figure 1):

$$D_\epsilon = S_\epsilon \cup S_{-\epsilon} \cup C_\epsilon,$$

where

$$\begin{aligned} S_\epsilon &= \{x \in \mathbb{R}^{s+1} : \sum_{i=1}^s x_i^2 + (x_{s+1} - \epsilon)^2 = 1, \wedge, x_{s+1} \geq \epsilon\}, \\ S_{-\epsilon} &= \{x \in \mathbb{R}^{s+1} : \sum_{i=1}^s x_i^2 + (x_{s+1} + \epsilon)^2 = 1, \wedge, x_{s+1} \leq -\epsilon\}, \\ C_\epsilon &= \{x \in \mathbb{R}^{s+1} : \sum_{i=1}^s x_i^2 = 1, \wedge, x_{s+1} \in [-\epsilon, \epsilon]\}. \end{aligned}$$

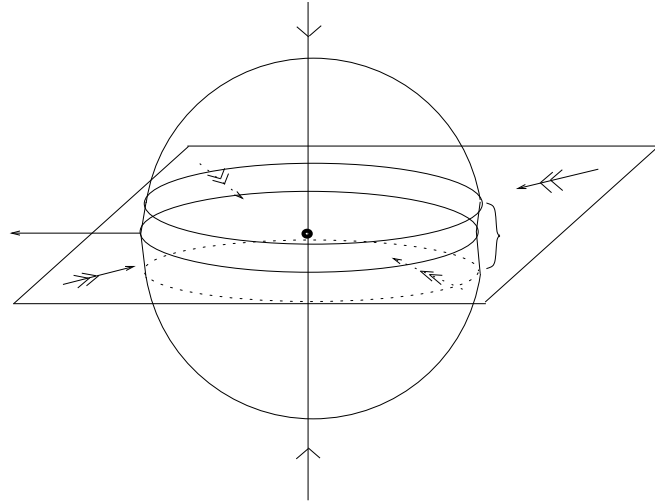


FIGURE 1. The fundamental domain.

As $W_{loc}^s(\sigma)$ is $(s+1)$ -dimensional and D_ϵ is homeomorphic to the sphere (s) -dimensional, by construction D_ϵ intersects $W_X^{ss}(\sigma)$ in $C_\epsilon|_{x_{s+1}=0}$ that is a sphere $(s-1)$ -dimensional. Note that the orbits of all point in $C_\epsilon|_{x_{s+1}=0}$ together with σ yields $W_X^{ss}(\sigma)$. In particular, $C_\epsilon|_{x_{s+1}=0} \notin M(X)$ by Theorem 3.3. Also note that for all ϵ , D_ϵ is a fundamental domain.

Let \widetilde{D}_ϵ be a cross section of X such that $W_{loc}^s(\sigma) \cap \widetilde{D}_\epsilon = D_\epsilon$. It follows that \widetilde{D}_ϵ is a $(n-1)$ -cylinder, and so we can consider a system coordinated (x, s) with $x \in D_\epsilon$ and $s \in I^u$. Thus, by using this system coordinate we can construct a family of singular cross-sections $\Sigma_\delta^t, \Sigma_\delta^b$ (for all $\delta \in [-\epsilon, \epsilon]$) by setting

$$\begin{aligned}\Sigma_\delta^t &= \{(x, s) \in \widetilde{D}_\epsilon : x \in S_\delta, s \in I^u\}, \\ \Sigma_\delta^b &= \{(x, s) \in \widetilde{D}_\epsilon : x \in S_{-\delta}, s \in I^u\}.\end{aligned}$$

Due to the smooth variation of $W_Y^{ss}(\sigma(Y))$ with respect to Y close to X we can assume that $\sigma(Y) = \sigma$ and that $W_{loc,Y}^{ss}(\sigma(Y)) = W_{loc}^{ss}(\sigma)$ for every Y close to X . By choosing D_ϵ so close to σ we can further assume that \widetilde{D}_ϵ is a cross-section of Y , for every Y close to X . We claim that there is $\delta > 0$ such that the conclusion of the lemma holds for $\Sigma^t = \Sigma_\delta^t$ and $\Sigma^b = \Sigma_\delta^b$. Indeed, we first note that under the cylindrical coordinate system (x, s) one has $\Sigma^{*,\Delta} = \Sigma_\Delta^*$ for all $0 < \Delta \leq \delta$ ($*$ = t, b). Otherwise, if the conclusion of the claim fails, it implies that O_n intersects $\widetilde{D}_\epsilon \setminus (\Sigma_\Delta^t \cup \Sigma_\Delta^b)$ for all $\Delta > 0$ small. Further, we would find a sequence of periodic points such that $p_n \in O_n$ (for all n large) and $p_n = (x_n, s_n)$ with $x_n \in C_\Delta$ and $s_n \rightarrow 0$ as $n \rightarrow \infty$. As Δ is arbitrary and $s_n \rightarrow 0$, we conclude that p_n converges to a point in $C_\Delta|_{x_{s+1}=0}$ by passing to a subsequence if necessary, since if $s_n \rightarrow 0$, it implies that the intersection tends to (s) -dimensional sphere D_ϵ .

As $O_n \subset M(Y^n)$, $Y^n \rightarrow X$ and $M(Y^n)$ is arbitrarily close to $M(X)$ (indeed, in the C^1 -topology, these sets are at least to a distance ϵ) for all n ($n \in \mathbb{N}$), by using the above arguments, there exists a point $z \in (C_\epsilon|_{x_{s+1}=0})$ such that $z \in M(X)$. This contradicts Theorem 3.3 and the proof follows. \square

4. Proof of Theorem 1.4

We prove the theorem by contradiction. Let X be a transitive sectional-Anosov flow C^1 of M . Then, we suppose that there exists a sequence $X^n \xrightarrow{C^1} X$ such that every X^n exhibits a non-singular attractor $A^n \in M(X^n)$ arbitrarily close to $M(X)$ and since A^n also is arbitrarily close to $M(X)$, we can assume that A^n belongs to $M(X)$ for all n . It follows from the definition of attractor that each A^n is compact, invariant and nonempty. As A^n is non-singular by hypothesis, then the Lemma 3.1 and the Lemma 3.2 imply the following:

$$\begin{aligned}A^n &\text{ is a hyperbolic attractor of type saddle of } X^n \text{ for all } n, \text{ and since} \\ A^n &\text{ is non-singular for all } n, \text{ obviously } A^n \text{ is not a singularity of } X^n \quad (3) \\ &\text{ for all } n.\end{aligned}$$

We denote by $Sing(X)$ the set of singularities of X , $Cl(A)$ the closure of A , $A \subset M$. Moreover, given $\delta > 0$ and $A \subset M$, we define $B_\delta(A) = \{x \in M : d(x, A) < \delta\}$ where $d(\cdot, \cdot)$ is the metric on M .

Now, let us consider the following lemma that, as in [9], is useful for the higher dimension case. The lemma gives one description on behavior of the attractors.

Lemma 4.1. *The sequence of attractors A^n accumulate on $Sing(X)$, i.e.*

$$Sing(X) \cap Cl\left(\bigcup_{n \in \mathbb{N}} A^n\right) \neq \emptyset.$$

Proof. We prove the lemma by contradiction. With that purpose we suppose that there is $\delta > 0$, such that

$$B_\delta(Sing(X)) \cap \left(\bigcup_{n \in \mathbb{N}} A^n\right) = \emptyset. \quad (4)$$

In the same way as in [9], we define

$$H = \bigcap_{t \in \mathbb{R}} X_t(M \setminus B_{\delta/2}(Sing(X))).$$

Note that by definition H is a invariant set and $Sing(X) \cap H = \emptyset$. Additionally H is a compact set as it is Λ and therefore H is a nonempty compact set (see [9]). By using the Lemma 3.1 we conclude that H is hyperbolic set. So, we denote by $E^s \oplus E^X \oplus E^u$ the corresponding hyperbolic splitting (see Definition 2.1).

By the stability of hyperbolic sets we can fix a neighborhood W of H and $\epsilon > 0$ such that if Y is a vector field C^r close to X and H_Y is a compact invariant set of Y in W then:

$$\begin{aligned} &H_Y \text{ is hyperbolic and its hyperbolic splitting } E^{s,Y} \oplus E^Y \oplus E^{u,Y}. \\ &\dim(E^u) = \dim(E^{u,Y}), \quad \dim(E^s) = \dim(E^{s,Y}). \text{ The manifolds } \\ &W_Y^{uu}(x, \epsilon), x \in H_Y, \text{ have uniform size } \epsilon. \end{aligned} \quad (5)$$

As $X^n \rightarrow X$, we have that:

$$\begin{aligned} &\bigcap_{t \in \mathbb{R}} X_t^n(M \setminus B_{\delta/2}(Sing(X))) \subset W, \text{ for all } n \text{ large;} \\ &A^n \subset M \setminus B_{\delta/2}(Sing(X)) \text{ for all } n, \text{ and } A^n \subset W \text{ for all } n \text{ large;} \\ &\text{If } x^n \in A^n \text{ so that } x^n \text{ converges to some } x \in M, \text{ then } x \in H; \\ &\text{If } w \in W_{X^n}^{uu}(x^n, \epsilon), \text{ the tangent vectors of } W_{X^n}^{uu}(x^n, \epsilon) \text{ in this} \\ &\text{point are in } E_w^{u, X^n}; \\ &\text{As } x^n \rightarrow x, W_{X^n}^{uu}(x^n, \epsilon) \rightarrow W_X^{uu}(x, \epsilon) \text{ in the sense of } C^1 \\ &\text{submanifolds [13] and } \angle(E^{u, X^n}, E^u) \rightarrow 0, \text{ if } n \rightarrow \infty \text{ [9].} \end{aligned} \quad (6)$$

Thus, we fix an open set $U \subset W_X^{uu}(x, \epsilon)$ containing the point x .

By (3), it follows that the periodic orbits of X^n in A^n are dense in A^n (Anosov closing Lemma). Particularly, we can assume that each x^n is a periodic point of A^n . As $M(X) \cap Sing(X) \neq \emptyset$ and $M(X)$ is a transitive set, it follows from Lemma 3.2 that there exists $q \in U$, $0 < \delta_1 < \delta_2 < \frac{\delta}{2}$ and $T > 0$ such that $X_T(q) \in B_{\delta_1}(Sing(X))$.

By [12, Tubular Flow Box Theorem], there is an open set V_q containing q such that $X_T(V_q) \subset B_{\delta_1}(Sing(X))$ and as $X^n \rightarrow X$ it follows that

$$X_T^n(V_q) \subset B_{\delta_2}(Sing(X)) \quad (7)$$

for all n large (see Figure 2).

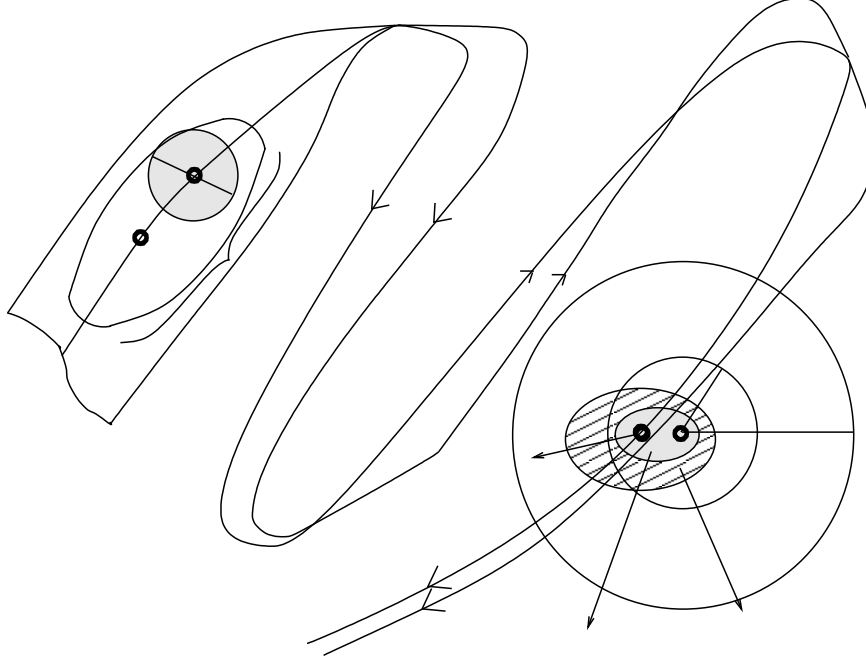


FIGURE 2. Tubular Flow Box Theorem for $X_T(V_q)$.

In addition, $W_{X^n}^{uu}(x^n, \epsilon) \cap V_q \neq \emptyset$ for n large enough, since $W_{X^n}^{uu}(x^n, \epsilon) \rightarrow W_X^{uu}(x, \epsilon)$ and $q \in U \subset W_X^{uu}(x, \epsilon)$. Applying (7) to X^n for n large we have

$$X_T^n(W_{X^n}^{uu}(x^n, \epsilon)) \cap B_{\delta_2}(Sing(X)) \neq \emptyset.$$

In particular $W_{X^n}^{uu}(x^n, \epsilon) \subset W_{X^n}^u(x^n)$. Then the invariance of $W_{X^n}^u(x^n)$ implies

$$W_{X^n}^u(x^n) \cap B_{\delta/2}(Sing(X)) \neq \emptyset.$$

Observe that $W_{X^n}^u(x^n) \subset A^n$ since $x^n \in A^n$ and A^n is an attractor. We conclude that

$$A^n \cap B_{\delta}(Sing(X)) \neq \emptyset.$$

This contradicts (4) and the proof follows. \square

Proof of Theorem 1.4. By Lemma 4.1 there exists $\sigma \in M(X)$ such that

$$\sigma \in \text{Sing}(X) \cap \text{Cl} \left(\bigcup_{n \in \mathbb{N}} A^n \right).$$

By Theorem 3.3 we have that σ is Lorenz-like and satisfies

$$M(X) \cap W_X^{ss}(\sigma) = \{\sigma\}.$$

By Proposition 3.6, we can choose Σ^t, Σ^b , singular-cross section for σ and $M(X)$ such that

$$M(X) \cap (\partial^h \Sigma^t \cup \partial^h \Sigma^b) = \emptyset.$$

As $X^n \rightarrow X$ we have that Σ^t, Σ^b is singular-cross section of X^n too, thus we can assume that $\sigma(X^n) = \sigma$ and $l^t \cup l^b \subset W_{X^n}^s(\sigma)$ for all n (Implicit function theorem).

We have that the splitting $E^s \oplus E^c$ persists by small perturbations of X [8]. The dominance condition (Definition 1.1-(2)) together with [6, Proposition 2.2] imply that for $* = t, b$ one has

$$T_x \Sigma^* \cap (E_x^s \oplus E_x^c) = T_x l^*,$$

for all $x \in l^*$.

Denote by $\angle(E, F)$ the angle between two linear subspaces. The last equality implies that there is $\rho > 0$ such that

$$\angle(T_x \Sigma^* \cap E_x^c, T_x l^*) > \rho,$$

for all $x \in l^*$ ($* = t, b$). In this way, since $E^{c,n} \rightarrow E^c$ as $n \rightarrow \infty$ we have for n large enough that

$$\angle(T_x \Sigma^* \cap E_x^{c,n}, T_x l^*) > \frac{\rho}{2}, \quad (8)$$

for all $x \in l^*$ ($* = t, b$).

As in the previous section we fix a coordinate system $(x, y) = (x^*, y^*)$ in Σ^* such that

$$\Sigma^* = B^u[0, 1] \times B^{ss}[0, 1], \quad l^* = \{0\} \times B^{ss}[0, 1]$$

with respect to (x, y) . Also, given $\Delta > 0$ we define $\Sigma^{*,\Delta} = B^u[0, \Delta] \times B^{ss}[0, 1]$.

Hereafter $\Pi^* : \Sigma^* \rightarrow B^u[0, 1]$ will be the projection such that $\Pi^*(x, y) = x$. We will denote the line field in Σ^{*,Δ_0} by F^n , where

$$F_x^n = T_x \Sigma^* \cap E_x^{c,n}, \quad x \in \Sigma^{*,\Delta_0}.$$

Remark 4.2. The continuity of $E^{c,n}$ and (8) imply that there is $\Delta_0 > 0$ such that for every n large the line F^n is *transverse to* Π^* . By this we mean that $F^n(z)$ is *not tangent to the curves* $(\Pi^*)^{-1}(c)$, for every $c \in B^u[0, \Delta_0]$.

Now recall that A^n is a hyperbolic attractor of type saddle of X^n for all n (see (3)) and that the periodic orbits of X^n in A^n are dense in A^n [13]. As $\sigma \in Cl(\cup_{n \in \mathbb{N}} A^n)$, we can find a sequence of periodic orbits $(O_n)_{n \in \mathbb{N}}$ such that $O_n \in A^n$ and accumulating on σ . It follows from Lemma 3.7 applied to $Y^n = X^n$ that there exists $n_0 \in \mathbb{N}$ such that either

$$O_{n_0} \cap \text{int}(\Sigma^{t, \Delta_0}) \neq \emptyset \quad \text{or} \quad O_{n_0} \cap \text{int}(\Sigma^{b, \Delta_0}) \neq \emptyset.$$

As $O_{n_0} \subset A_{n_0}$ we conclude that either

$$A^{n_0} \cap \text{int}(\Sigma^{t, \Delta_0}) \neq \emptyset \quad \text{or} \quad A^{n_0} \cap \text{int}(\Sigma^{b, \Delta_0}) \neq \emptyset.$$

We shall assume that $A^{n_0} \cap \text{int}(\Sigma^{t, \Delta_0}) \neq \emptyset$ (Analogous proof for the case $* = b$). Note that $\partial^h \Sigma^{t, \Delta_0} \subset \partial^h \Sigma^t$ by definition. Then, by Proposition 3.6 one has

$$A \cap \partial^h \Sigma^{t, \Delta_0} = \emptyset.$$

As A^{n_0} and Σ^{t, Δ_0} are compact non-empty sets, it follows that $A^{n_0} \cap \Sigma^{t, \Delta_0}$ is a compact nonempty subset of Σ^{t, Δ_0} , and thus there exists $p \in \Sigma^{t, \Delta_0} \cap A^{n_0}$ such that

$$\text{dist}(\Pi^t(\Sigma^{t, \Delta_0} \cap A^{n_0}), 0) = \text{dist}(\Pi^t(p), 0),$$

where dist denotes the distance in $B^u[0, \Delta_0]$. Note that $\text{dist}(\Pi^t(p), 0)$ is the minimum distance of $\Pi^t(\Sigma^{t, \Delta_0} \cap A^{n_0})$ to 0 in $B^u[0, \Delta_0]$.

As $p \in A^{n_0}$, we have that $W_{X^{n_0}}^u(p)$ is a well defined submanifold, since that A^{n_0} is hyperbolic set (see (3)), and $\dim(E^c) = \dim(E^{c, n_0})$ (see (5)).

By domination Definition 1.1-(2), $T_z(W_{X^{n_0}}^u(p)) = E_z^{c, n_0}$ for every $z \in W_{X^{n_0}}^u(p)$ and hence $\dim(W_{X^{n_0}}^u(p)) = (n - s - 1)$ (2). Next, we can ensure that

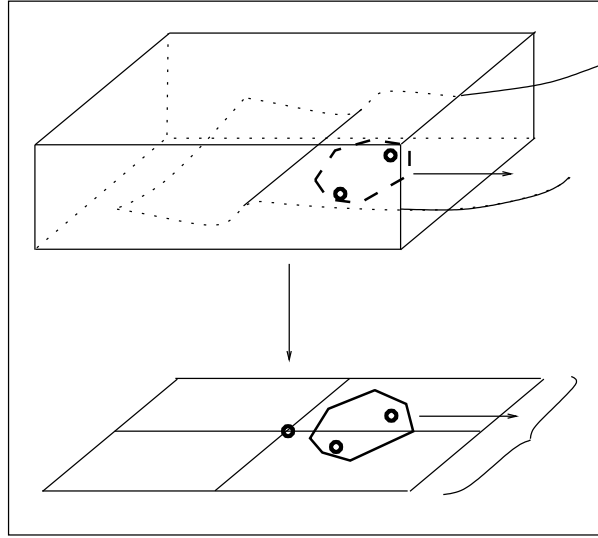
$$T_z(W_{X^{n_0}}^u(p)) \cap T_z \Sigma^{t, \Delta_0} = E_z^{c, n_0} \cap T_z \Sigma^{t, \Delta_0} = F_z^{n_0}$$

for every $z \in W_{X^{n_0}}^u(p) \cap \Sigma^{t, \Delta_0}$.

Note that the last equality shows that $W_{X^{n_0}}^u(p) \cap \Sigma^{t, \Delta_0}$ is transversal, and therefore there exists some compact submanifold inside of $W_{X^{n_0}}^u(p) \cap \Sigma^{t, \Delta_0}$. We denote this compact submanifold by K^{n_0} . Thus by construction $p \in K^{n_0}$ (see (3)) and K^{n_0} is tangent to F^{n_0} , since $K^{n_0} \subset W_{X^{n_0}}^u(p) \cap \Sigma^{t, \Delta_0}$.

Remark 4.3. By construction we have that $\dim(B^u[0, \Delta_0]) = (n - s - 1)$, since $\dim(E^{c, n_0}) = \dim(W_{X^{n_0}}^u(p)) = (n - s - 1)$.

We have that $W_{X^{n_0}}^u(p) \cap \Sigma^{t, \Delta_0}$ is a submanifold of M , since $W_{X^{n_0}}^u(p) \cap \Sigma^{t, \Delta_0}$ is transversal and nonempty and $W_{X^{n_0}}^u(p)$, Σ^{t, Δ_0} are submanifolds of M . Note that $\dim(W_{X^{n_0}}^u(p)) + \dim(\Sigma^{t, \Delta_0}) \geq n$.

FIGURE 3. The projection $\Pi^t(K^{n_0}) = K_1^{n_0}$.

Since F^{n_0} is *transverse* to Π^t , one has that K^{n_0} is *transverse* to Π^t (i.e. K^{n_0} is transverse to the curves $(\Pi^t)^{-1}(c)$, for every $c \in B^u[0, \Delta_0]$). Let us denote the image of K^{n_1} by the projection Π^t in $B^u[0, \Delta_0]$ by $K_1^{n_1}$, i.e., $\Pi^t(K^{n_1}) = K_1^{n_1}$. Note that $K_1^{n_1} \subset B^u[0, \Delta_0]$ and $\Pi^t(p) \in \text{int}(K_1^{n_1})$ (See Figure 3).

As $\dim(K_1^{n_0}) = \dim(B^u[0, \Delta_0])$ (By Remark 4.3), there exists $z_0 \in K^{n_0}$ such that

$$\text{dist}(\Pi^t(z_0), 0) < \text{dist}(\Pi^t(p), 0).$$

It follows from the property of attractor that $W_{X^{n_0}}^{uu}(p, \epsilon) \subset W_{X^{n_0}}^u(p) \subset A^{n_0}$. Thus, $K^{n_0} \subset \Sigma^{t, \Delta_0} \cap A^{n_0}$ and $p \in A^{n_0}$.

Since, by Proposition 3.6 $A^{n_0} \cap \partial^h \Sigma^{t, \Delta_0} = \emptyset$ and, by Remark 4.3 $\dim(K_1^{n_0}) = \dim(B^u[0, \Delta_0])$, we conclude that

$$\text{dist}(\Pi^t(\Sigma^{t, \Delta_0} \cap A^{n_0}), 0) = 0.$$

Given that A^{n_0} is closed, this last equality implies

$$A^{n_0} \cap l^t \neq \emptyset.$$

Since $l^t \subset W_{X^{n_0}}^s(\sigma)$ and A^{n_0} is closed invariant set for X^{n_0} we conclude that $\sigma \in A^{n_0}$. We have proved that A^{n_0} contains a singularity of X^{n_0} . But A^{n_0} is a hyperbolic attractor of X^{n_0} by the Property (3) and this leads to $A^{n_0} = \{\sigma\}$. Finally, using the Property (3) we obtain a contradiction and the proof follows. \checkmark

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