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Generalized Rigid Modules

Módulos generalizados rígidos

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Abstract. Let α be an endomorphism of an arbitrary ring R with identity. The aim of this paper is to introduce the notion of an α-rigid module which is an extension of the rigid property in rings and the α-reduced property in modules defined in [8]. The class of α-rigid modules is a new kind of modules which behave like rigid rings. A right R-module M is called *α-rigid* if $ma\alpha(a) = 0$ implies $ma = 0$ for any $m \in M$ and $a \in R$. We investigate some properties of α-rigid modules and among others we also prove that if $M[x;\alpha]$ is a reduced right $R[x;\alpha]$-module, then M is an α-rigid right R-module. The ring R is α-rigid if and only if every flat right R-module is α-rigid. For a rigid right R-module M, M is α-semicommutative if and only if $M[x;\alpha]_{R[x;\,\alpha]}$ is semicommutative if and only if $M\big[[x;\alpha]\big]_{R[[x;\,\alpha]]}$ is semicommutative.

Key words and phrases. Reduced modules, Semicommutative modules, Armendariz modules, Rigid modules.

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Resumen. Sea α un endomorfismo de un anillo arbitrario R con identidad. El propósito de este articulo es introducir la noción de un módulo α-rígido el cual es una extensión de la propiedad de rigidez en anillos y la propiedad de α-reducibilidad en módulos definida en [8]. La clase de módulos α-rígidos es una nueva clase de módulos los cuales de comportan como anillos rígidos. Un R-módulo derecho M es llamado *α-rígido* si $ma\alpha(a) = 0$ implica que $ma = 0$ para cualquier $m \in M$ y $a \in R$. Nosotros investigamos algunas propiedades de módulos α-rígidos y entre otras nosotros también probamos que si $M[x;\alpha]$ es un $R[x;\alpha]$-módulo derecho reducido, entonces M es un R-módulo derecho α-rígido. El anillo R es α-rígido si y sólo si cada R-módulo bandera derecha es α-rígido. Para un R-módulo derecho rígido M, M es α-semiconmutativo si y sólo si $M[x;\alpha]_{R[x;\,\alpha]}$ es semiconmutativo si y sólo si $M\big[[x;\alpha]\big]_{R[[x;\,\alpha]]}$ es semiconmutativo.

Palabras y frases clave. Módulos reducidos, módulos semiconmutativos, módulos de Armendariz, módulos rigidos.

1. Introduction

Throughout this paper R denotes a ring with identity, modules are unital right R-modules and all ring homomorphisms are unital (unless explicitly stated otherwise). The letter α denotes an endomorphism of R and it will be fixed in the sequel. Let M be a right R-module. The module M is called *α-compatible* if for any $m \in M$ and any $a \in R$, $ma = 0$ if and only if $m\alpha(a) = 0$. An α-compatible module M is called *α-reduced* if for any $m \in M$ and any $a \in R$, $ma = 0$ implies $mR \bigcap Ma = 0$. The module M is called *reduced* if it is $\mathbf{1}$-reduced where $\mathbf{1}$ denotes the identity endomorphism of M. Hence a ring R is reduced (i.e., has no nonzero nilpotent elements) if and only if the right R-module R_R is a reduced module. In [9], Rege and Chhawchharia introduced the notion of an Armendariz ring. A ring R is called *Armendariz* if for any $f(x) = \sum\limits_{i=0}^{n} a_i x^i$, $g(x) = \sum\limits_{j=0}^{s} b_j x^j \in R[x]$, $f(x)g(x) = 0$ implies that $a_i b_j = 0$ for all i and j.

We write $R[x]$, $R\big[[x]\big]$, $R\big[x, x^{-1}\big]$ and $R\big[[x, x^{-1}]\big]$ for the polynomial ring, the power series ring, the Laurent polynomial ring and the Laurent power series ring over R, respectively. Similarly, $R[x;\alpha]$, $R\big[[x;\alpha]\big]$, $R\big[x, x^{-1};\alpha\big]$ and $R\big[[x, x^{-1};\alpha]\big]$ denote the skew polynomial ring, the skew power series ring, the skew Laurent polynomial ring and the skew Laurent power series ring over R, respectively. Following Lee and Zhou [8] for a right R-module M, we write

$$M[x;\alpha] = \left\{ \sum_{i=0}^{s} m_i x^i : s \geq 0, m_i \in M \right\},$$

$$M\big[[x;\alpha]\big] = \left\{ \sum_{i=0}^{\infty} m_i x^i : m_i \in M \right\},$$

$$M\big[x, x^{-1};\alpha\big] = \left\{ \sum_{i=-s}^{t} m_i x^i : s \geq 0, t \geq 0, m_i \in M \right\},$$

$$M\big[[x, x^{-1};\alpha]\big] = \left\{ \sum_{i=-s}^{\infty} m_i x^i : s \geq 0, m_i \in M \right\}.$$

Each of these is an abelian group under an obvious addition operation. Moreover $M[x;\alpha]$ becomes a right module over $R[x;\alpha]$ under the following scalar product operation:

$$m(x)f(x) = \sum_{k=0}^{s+t} \left(\sum_{i+j=k} m_i \alpha^i(a_j) \right) x^k,$$

where $m(x) = \sum_{i=0}^{s} m_i x^i \in M[x;\alpha]$ and $f(x) = \sum\limits_{i=0}^{t} a_i x^i \in R[x;\alpha]$. Similarly, $M\big[[x;\alpha]\big]$ is a module over $R\big[[x;\alpha]\big]$. The modules $M[x;\alpha]$ and $M\big[[x;\alpha]\big]$

are called, respectively, the *skew polynomial extension* and the *skew power series extension* of M. If $\alpha \in Aut(R)$, then with a similar scalar product, $M\left[[x, x^{-1}; \alpha]\right]$ (resp. $M\left[x, x^{-1}; \alpha\right]$) becomes a module over $R\left[[x, x^{-1}; \alpha]\right]$ (resp. $R[x, x^{-1}; \alpha]$). The modules $M\left[x, x^{-1}; \alpha\right]$ and $M\left[[x, x^{-1}; \alpha]\right]$ are called the *skew Laurent polynomial extension* and the *skew Laurent power series extension* of M, respectively.

In [7], an endomorphism α of a ring R is said to be *rigid* if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$, and R is said to be an *α-rigid ring* if the endomorphism α is rigid. Recently, rigid rings are generalized to central rigid rings by Ungor et. al. in [6]. A ring R is called *central rigid* if for any $a, b \in R$, $a^2b = 0$ implies ab is central. Motivated by the results and properties of α-rigid rings that have been studied in Hirano [2], Hong et. al. [3, 4] and Krempa [7], we introduce and study a generalization of α-rigid rings which we call α-rigid modules. We investigate some properties of α-rigid modules, and the relations between α-rigid modules, α-semicommutative modules and extended Armendariz modules. Every α-reduced module is α-rigid but the converse need not be true in module case in general, however it is true for rings, that is, a ring is α-reduced if and only if it is α-rigid (Corollary 2.17). We supply an example to show that there exists an α-rigid module which is not α-reduced (Example 2.18). For a right R-module M, we prove that if $M[x; \alpha]$ is a reduced right $R[x; \alpha]$-module, then M is an α-rigid right R-module (Theorem 2.19). We also show that if M is an α-semicommutative and rigid module, then M is α-skew Armendariz, in particular, if M is semicommutative and rigid, then M is Armendariz (Theorem 2.28). We combine α-rigid modules with α-semicommutative modules and α-Armendariz modules and prove that (i) If M is an α-Armendariz right R-module, then M is α-semicommutative if and only if $M[x; \alpha]$ is semicommutative as an $R[x; \alpha]$-module. (ii) An Armendariz right R-module M is α-semicommutative if and only if $M[x]$ is an α-semicommutative right $R[x]$-module.

2. Generalized Rigid Modules

In this section, we introduce the concept of an α-rigid module which is an extension of the rigid property in rings. We investigate some properties of α-rigid modules and α-semicommutative modules.

We start with the following definition.

Definition 2.1. Let M be a right R-module. We call M an *α-rigid module* if $ma\alpha(a) = 0$ implies $ma = 0$ for any $m \in M$ and $a \in R$. The module M is called *rigid* if it is $\mathbf{1}$-rigid.

By Definition 2.1, it is clear that a ring R is α-rigid if and only if the right R-module R_R is α-rigid.

We mention some classes of rigid modules.

Examples 2.2.

(1) Every α-reduced module is α-rigid.

(2) Let p be a prime number, n a positive integer and $\mathbb{Z}_{p^n}$ the ring of integers modulo p^n. Then $p^{n-1}\mathbb{Z}_{p^n}$ is a rigid module over $\mathbb{Z}_{p^n}$.

(3) All rigid modules are reduced over von Neumann regular rings.

(4) Semiprime rigid modules are reduced.

Lemma 2.3.

(1) The class of α-rigid modules is closed under submodules, direct products, and therefore under direct sums.

(2) Let $\{M_i : i \in I\}$ be a class of modules. Then each $M_i (i \in I)$ is α-rigid if and only if $\prod_{i\in I} M_i$ is α-rigid if and only if $\bigoplus_{i\in I} M_i$ is α-rigid.

Proof. Clear from the definitions. ☑

Proposition 2.4. *Let M be an α-rigid module and let $a \in R$ satisfying $\alpha(a^2) = 0$. Then $Ma = 0$.*

Proof. For any $m \in M$, we have $ma\alpha(a)\alpha\big(a\alpha(a)\big) = ma\alpha(a^2)\alpha^2(a) = 0$. By hypothesis, $ma\alpha(a) = 0$ which implies $ma = 0$. ☑

Corollary 2.5. *Let R be an α-rigid ring. Then α is a monomorphism and R is a reduced ring.*

Note that the class of α-rigid modules is not closed under extensions.

Example 2.6. Let F be a field and R the upper triangular matrix ring $\begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ over F, $\alpha : R \to R$ given by $\alpha \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} a & -b \\ 0 & c \end{bmatrix}$, the right R-module $M = \begin{bmatrix} 0 & F \\ F & F \end{bmatrix}$, the submodule $K = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$. It is easy to check that K and M/K are α-rigid modules. The element $a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ satisfies $a^2 = 0$, $Ma \neq 0$. By Proposition 2.4, M is not α-rigid.

The ring R is called *semicommutative* if whenever $a, b \in R$ satisfy $ab = 0$, we have $aRb = 0$. In Definition 2.7 we recall an extension of this definition to modules. Agayev and Harmanci investigated α-semicommutative rings and modules in [1] and focused on the semicommutativity of subrings of matrix rings. In this note we continue the study of α-semicommutative modules.

Definition 2.7. Let M be a right R-module. Then M is called *α-semicommutative* if for any $m \in M$ and any $a \in R$, $ma = 0$ implies $mR\alpha(a) = 0$. The module M is called *semicommutative* if it is $\mathbf{1}$-semicommutative. The ring R is said to be *α-semicommutative* if the right R-module R_R is α-semicommutative.

We observe the following lemma for future use.

Lemma 2.8. *Let M be an α-semicommutative module. For any $m \in M$ and any $a \in R$, we have*

(1) $m\alpha^i(a) = 0$ implies $mR\alpha^{i+1}(a) = 0$, for all positive integer i.

(2) $ma^2 = 0$ implies $ma\alpha(a) = 0$.

Proof. The proof is straightforward. ☑

Next we show that α-semicommutativity of modules is not preserved under extensions.

Example 2.9. Let R, M, K and α be as in Example 2.6. It easy to check that K and M/K are α-semicommutative modules. Let $m = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \in M$ and $r = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, $a = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \in R$. Then $ma = 0$. But $mr\alpha(a) \neq 0$. Hence M is not α-semicommutative.

Next we recall a condition which ensures that the α-semicommutativity of the modules N and M/N (for a submodule N of M) implies that of M. A submodule N of a module M is called *prime submodule* of M if whenever $ma \in N$ for $m \in M$ and $a \in R$, then $m \in N$ or $Ma \subseteq N$.

Proposition 2.10. *Let M be a right R-module with a submodule N. Assume that M/N and N are α-semicommutative modules. If N is a prime submodule of M and M/N is torsion-free, then M is α-semicommutative.*

Proof. Let $ma = 0$ for some $m \in M$ and $a \in R$. Then $ma \in N$. If $m \in N$, then $mR\alpha(a) = 0$ since N is α-semicommutative. Assume that $m \notin N$. Hence $\overline{m}$ is nonzero in M/N. By hypothesis, $\overline{m}a = 0$ implies that $\overline{m}R\alpha(a) = 0$. Hence $mR\alpha(a) \subseteq N$. It follows that $M\alpha(a) \subseteq N$ because N is prime. That means $(M/N)\alpha(a) = 0$. Hence $\alpha(a) = 0$ since M/N is torsion-free. This completes the proof. ☑

There are semicommutative modules which are not α-semicommutative.

Example 2.11. Let S be a commutative nonzero ring and $R = S \times S$. Let M be the R-module R_R. Let $\alpha : R \to R$ be defined by $\alpha(a, b) = (b, a)$. Then α is an automorphism of R. Let $(m_1, m_2) \in M$ and $(a_1, a_2), (b_1, b_2) \in R$ and suppose $(m_1, m_2)(a_1, a_2) = (m_1 a_1, m_2 a_2) = 0$. Since R is commutative, we have $(m_1, m_2)(b_1, b_2)(a_1, a_2) = (m_1 b_1 a_1, m_2 b_2 a_2) = (m_1 a_1 b_1, m_2 a_2 b_2) = 0$. Then M is semicommutative. For $(1, 0) \in M$ and $(0, 1) \in R$, we have $(1, 0)(0, 1) = 0$ but $(1, 0)R\alpha(0, 1) \neq 0$. Hence M is not α-semicommutative.

An example of a unital endomorphism α of a nonsemicommutative ring R which makes it α-semicommutative exists in the literature [11, Example 2.4]. Example 2.12 does the same for a nonunital endomomorphism α.

Example 2.12. Let F be any field and e_{ij} denote the 3×3 matrix units. Let $R = \begin{bmatrix} F & 0 & F \\ 0 & F & 0 \\ 0 & 0 & F \end{bmatrix}$, let $M = \begin{bmatrix} 0 & 0 & F \\ 0 & F & 0 \\ F & 0 & F \end{bmatrix}$ be a right R-module and let $\alpha : R \to R$, $\alpha(a_{ij}) = e_{22}a_{22}$, where $(a_{ij}) \in R$. Then α is a well-defined endomorphism of R and which is not unital. Let $m = e_{31} + e_{33} \in M$, $a = e_{13} - e_{33}$ and $r = e_{13}u + e_{33}v \in R$, where $u, v \in F$ with $u + v \neq 0$. Then $ma = 0$ but $mra \neq 0$. Therefore M is not a semicommutative as an R-module.

Now let $m = (m_{ij}) \in M$ and $a = (a_{ij}) \in R$ with $ma = 0$. Then we have

$$m_{13}a_{33} = 0 \tag{1}$$
$$m_{22}a_{22} = 0 \tag{2}$$
$$m_{31}a_{11} = 0 \tag{3}$$
$$m_{31}a_{13} + m_{33}a_{33} = 0 \tag{4}$$

If $n = (n_{ij}) \in R$, then $mn\alpha(a) = e_{22}m_{22}n_{22}a_{22} = 0$ from (2) above. Hence M is α-semicommutative.

However note the following fact.

Proposition 2.13. *Let M be an α-semicommutative and α-rigid module. Then M is semicommutative.*

Proof. Let $m \in M$ and $a \in R$ such that $ma = 0$. Since M is α-semicommutative, $mR\alpha(a) = 0$. For any $r \in R$, we have $mra\alpha(r)\alpha(a) = 0$. Hence $m(ra)\alpha(ra) = 0$. The module M being α-rigid $mra = 0$. It follows that $mRa = 0$. This completes the proof. ☑

A *regular element* of a ring R is an element which is not a zero divisor. Let S be a multiplicatively closed subset of R consisting of all central regular elements. We may localize R and M at S and ask when the localization $S^{-1}M$ is α-semicommutative as a right $S^{-1}R$-module. If $\alpha : R \to R$ is an endomorphism of the ring R, then $S^{-1}\alpha : S^{-1}R \to S^{-1}R$ defined by $S^{-1}\alpha(a/s) = \alpha(a)/s$ is an endomorphism of the ring $S^{-1}R$. Clearly this map extends α and we shall also denote this extended map by α.

Proposition 2.14. *Let S be a multiplicatively closed subset of R consisting of all central regular elements. An right R-module M is α-semicommutative if and only if the right $S^{-1}R$-module $S^{-1}M$ is α-semicommutative.*

Proof. If the right $S^{-1}R$-module $S^{-1}M$ is α-semicommutative, then it is clear that M is an α-semicommutative right R-module. Assume that M is α-semicommutative and $(m/s)(a/t) = 0$ in $S^{-1}M$, where $(m/s) \in S^{-1}M$, $a/t \in S^{-1}R$. So $ma = 0$. By assumption, $mR\alpha(a) = 0$. It implies $(m/s)(r/u)\big(\alpha(a)/t\big) = 0$ for all $r/u \in S^{-1}R$. Then $(m/s)S^{-1}R\big(\alpha(a)/t\big) = 0$ and so $S^{-1}M$ is α-semicommutative. ☑

Corollary 2.15. *Let M be a right R-module. The right $R[x]$-module $M[x]$ is α-semicommutative if and only if the right $R\big[x, x^{-1}\big]$-module $M\big[x, x^{-1}\big]$ is α-semicommutative.*

Proof. Let $S = \{1, x, x^2, \ldots\}$. Then S is a multiplicatively closed subset consisting of all central regular elements of $R[x]$. Since $S^{-1}M[x] = M\big[x, x^{-1}\big]$ and $S^{-1}R[x] = R\big[x, x^{-1}\big]$, the result is clear from Proposition 2.14. ☑

Theorem 2.16. *Let M be a right R-module. If M is an α-reduced module, then M is an α-rigid module. The converse holds if M is an α-semicommutative module.*

Proof. Let M be a right R-module. By [8, Lemma 1.2(2)(a)], M is an α-reduced module if and only if for any $m \in M$ and any $a \in R$,

(i) $ma = 0$ implies $mR\alpha(a) = 0$, and

(ii) $ma\alpha(a) = 0$ implies $ma = 0$.

Therefore M is an α-reduced module if and only if M is α-rigid and α-semicommutative. ☑

Corollary 2.17 is just mentioned and proved implicitly in [3]. But we give the proof for the sake of completeness.

Corollary 2.17. *Let α be an endomorphism of a ring R. Then R is α-reduced if and only if R is α-rigid.*

Proof. Let R be an α-reduced ring. By Theorem 2.16, R is α-rigid. Conversely, assume that $ab = 0$, for any $a, b \in R$. By Corollary 2.5, R is reduced and so $aR \bigcap Rb = 0$. Next we prove that $ab = 0$ if and only if $a\alpha(b) = 0$. If $ab = 0$, then $\big[a\alpha(b)\big]\big[\alpha\big(a\alpha(b)\big)\big] = a\alpha(b)\alpha(a)\alpha^2(b) = a\alpha(ba)\alpha^2(b) = 0$. Since R is α-rigid, then $a\alpha(b) = 0$. Conversely, suppose $a\alpha(b) = 0$. Multiplying from the right by $\alpha(a)$ and from the left by b, we have $ba\alpha(ba) = 0$. By assumption $ba = 0$, and then $ab = 0$. Therefore R is α-reduced. ☑

The next example shows that the converse implication of the first statement in Theorem 2.16 is not true in general, i.e., there exists a rigid module which is not reduced.

Example 2.18. Let $R = F\langle x, y\rangle$ be the ring of polynomials in two non-commuting indeterminates over a field F. Since the right ideal xR is not two-sided, the right R-module $M = R/xR$ is not semicommutative and so it is not reduced. Let $\overline{0} \neq \overline{m} = m + xR \in M$, where $m \in R$. Assume that $\overline{m}f^2 = 0$ for a nonzero $f \in R$. Then $mf^2 \in xR$. We express mf in the form $a + xg + yh$, where $a \in F, g, h \in R$. We have $af + xgf + yhf = mf^2 \in xR$. Hence $hf = 0$ yielding (since $f \neq 0$) $h = 0$. Further, if $a \neq 0$ then $a = mf - xg$ implies both m and f have nonzero constant terms. However, the element mf^2 in xR has no nonzero constant term, so $a = 0$ necessarily. Hence $mf = xg \in xR$ yielding $\overline{m}f = 0$. Thus M is a rigid module.

Theorem 2.19. *Let M be a right R-module. If $M[x;\alpha]$ is a reduced $R[x;\alpha]$-module, then M is an α-rigid R-module.*

Proof. If $M[x;\alpha]$ is reduced, then by [8, Theorem 1.6], M is α-reduced. Hence M is α-rigid from Theorem 2.16. ☑

Theorem 2.20. *A ring R is α-rigid if and only if every flat right R-module is α-rigid.*

Proof. Assume that R is α-rigid. Let M be a flat R-module and $m \in M$, $r \in R$ with $mr\alpha(r) = 0$. We prove $mr = 0$. Let $0 \to K \to F \to M \to 0$ be an exact sequence with F free and $M = F/K$. We write $m = y + K$ for $m \in M$ and $y \in F$. Then $yr\alpha(r) \in K$. By Villamayor Theorem [10, Theorem 3.62] there exists an homomorphism $f : F \to K$ so that $f\big(yr\alpha(r)\big) = yr\alpha(r)$. Write $u = f(y) - y$. Then $u \in F$ and $ur\alpha(r) = 0$. Since F is an α-rigid module as a direct sum copies of the α-rigid ring R, $ur = 0$, it follows that $f(y)r = yr$. Since $f(y)r = yr \in K$, $mr = y + K = 0$ in M. The other implication is trivial since R_R is a flat R-module. ☑

***Remark* 2.21.** Let S be a subring of a ring R with $1_R \in S$, $\alpha \in \mathrm{End}(R)$ such that $\alpha(S) \subseteq S$. Assume that a right S-module M_S is contained in a right R-module L_R. If L is α-semicommutative as an R-module, then M is α-semicommutative as an S-module.

Let M be a right R-module. We now determine the conditions under which the skew (Laurent) polynomial extension and the skew (Laurent) power series extension of the module M are semicommutative.

Lemma 2.22. *Let M be a right R-module. If M is an α-semicommutative and rigid module, $m(x) = \sum_{i=0}^{\infty} m_i x^i \in M\big[[x;\alpha]\big]$ and $f(x) = \sum_{j=0}^{\infty} a_j x^j \in R\big[[x;\alpha]\big]$. If $m(x)f(x) = 0$, then $m_i\alpha^i(a_j) = 0$ for all i and j.*

Proof. If M is an α-semicommutative module, then for any $m \in M$ and any $a \in R$, $ma = 0$ implies $mR\alpha(a) = 0$. By the definition of a rigid module, for any $m \in M$ and $a \in R$, $ma^2 = 0$ implies $ma = 0$. Therefore, the proof is similar to that of [8, Lemma 1.5]. ☑

The proof of Lemma 2.23 is the same as that of (2) ⇒ (1) of Proposition 3.1 of [11].

Lemma 2.23. *Let M be a right R-module. If $M[x;\alpha]$ is a semicommutative $R[x;\alpha]$-module, then M is α-semicommutative and semicommutative.*

Theorem 2.24. *Let M be a rigid right R-module. Then the following are equivalent:*

(1) M is α-semicommutative.

(2) $M[x;\alpha]_{R[x;\alpha]}$ is semicommutative.

(3) $M\big[[x;\alpha]\big]_{R[[x;\ \alpha]]}$ is semicommutative.

If $\alpha \in Aut(R)$, then the conditions (1)–(3) are equivalent to each of following:

(4) $M[x,x^{-1};\alpha]_{R[x,x^{-1};\alpha]}$ is semicommutative.

(5) $M\big[[x,x^{-1};\alpha]\big]_{R[[x,x^{-1};\alpha]]}$ is semicommutative.

Proof. By Remark 2.21, (5) ⇒ (3) ⇒ (2) and (5) ⇒ (4) ⇒ (2) are clear.

(1) ⇒ (3) Let M be an α-semicommutative module and assume that $m(x)f(x) = 0$, where $m(x) = \sum_{i=0}^{\infty} m_i x^i \in M\big[[x;\alpha]\big]$ and $f(x) = \sum_{j=0}^{\infty} a_j x^j \in R\big[[x;\alpha]\big]$. We now prove that $m(x)R\big[[x;\alpha]\big]f(x) = 0$. So, it suffices to show that $m(x)g(x)f(x) = 0$ for $m(x) = \sum_{i=0}^{\infty} m_i x^i \in M\big[[x;\alpha]\big]$ and $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $g(x) = \sum_{j=0}^{\infty} b_j x^j \in R\big[[x;\alpha]\big]$. By Lemma 2.22, we have $m_i\alpha^i(a_j) = 0$, for all i and j. Since M is α-semicommutative, by Lemma 2.8 $m_iR\alpha^{i+k}(a_j) = 0$, for all i, j, k. Then

$$\begin{aligned} m(x)g(x)f(x) &= \left(\sum_{i=0}^{\infty} m_i x^i\right)\left(\sum_{j=0}^{\infty} b_j x^j\right)\left(\sum_{k=0}^{\infty} a_k x^k\right) \\ &= \sum_i \sum_j \sum_k \left(m_i x^i\right)\left(b_j x^j\right)\left(a_k x^k\right) \\ &= \sum_i \sum_j \sum_k m_i \alpha^i(b_j)\alpha^{i+j}(a_k) x^{i+j+k} = 0 \end{aligned}$$

therefore $M\left[[x;\alpha]\right]_{R[[x;\alpha]]}$ is semicommutative.

(2) $\Rightarrow$ (1) Clear from Lemma 2.23.

(3) $\Rightarrow$ (5) Let $m(x)f(x) = 0$, where $m(x) = \sum\limits_{i=-k}^{\infty} m_i x^i \in M\left[[x,x^{-1};\alpha]\right]_{R[[x;\alpha]]}$ and $f(x) = \sum\limits_{j=-k}^{\infty} a_j x^j \in R\left[[x,x^{-1};\alpha]\right]$. In order to prove $m(x)R\left[[x,x^{-1};\alpha]\right]f(x) = 0$, it suffices to show that $m(x)g(x)f(x) = 0$ for any $g(x) = \sum\limits_{l=-k}^{\infty} b_l x^l \in R\left[[x,x^{-1};\alpha]\right]$. There exists $k > 0$ such that $m(x)x^k \in M\left[[x;\alpha]\right]$, $\left[\left(\sum\limits_{l=-k}^{\infty} \alpha^{-k}(a_l)x^l\right)x^k\right] \in R\left[[x;\alpha]\right]$. But

$$\left[m(x)x^k\right]\left[\left(\sum_{l=-k}^{\infty} \alpha^{-k}(a_l)x^l\right)x^k\right] = m(x)f(x)x^{2k} = 0.$$

Since $M\left[[x;\alpha]\right]_{R[[x;\alpha]]}$ is semicommutative,

$$\left[m(x)x^k\right]R\left[[x;\alpha]\right]\left[\left(\sum_{l=-k}^{\infty} \alpha^{-k}(a_l)x^l\right)x^k\right] = 0.$$

Hence

$$\left[m(x)x^k\right]\left[\left(\sum_{j=-k}^{\infty} \alpha^{-k}(b_j)x^j\right)x^k\right]\left[\left(\sum_{l=-k}^{\infty} \alpha^{-k}(a_l)x^l\right)x^k\right] = m(x)g(x)f(x)x^{3k} = 0.$$

So $m(x)g(x)f(x) = 0$ and $M\left[[x,x^{-1};\alpha]\right]_{R[[x,x^{-1};\alpha]]}$ is semicommutative. ☑

Corollary 2.25. *Let M be a rigid module. Then M is semicommutative if and only if so is $M\left[[x,x^{-1}]\right]$.*

Hong, Kwak and Rizvi [5] gave a generalization of Armendariz rings. Let α be an endomorphism of the ring R. R is called an *α-Armendariz ring* if for any $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{s} b_j x^j \in R[x;\alpha]$, $f(x)g(x) = 0$ implies $a_i b_j = 0$ for all i and j. Lee and Zhou extended the concept of α-Armendariz ring to modules in [8] so that Armendariz rings are generalized to modules. An α-compatible module M is called *α-Armendariz* if for any $m(x) = \sum_{i=0}^{n} m_i x^i \in M[x;\alpha]$ and $f(x) = \sum_{j=0}^{s} a_j x^j \in R[x;\alpha]$, $m(x)f(x) = 0$ implies $m_i\alpha^i(a_j) = 0$ for all i and j. The module M is called *Armendariz* if it is $\mathbf{1}$-Armendariz. Let α be an endomorphism of R. By definition $\alpha\big(f(x)\big) = \sum_{j=0}^{s} \alpha(a_j)x^j$ for $f(x) = \sum_{j=0}^{s} a_j x^j \in R[x]$, α extends also an endomorphism of $R[x]$.

All reduced rings are Armendariz. There are non-reduced Armendariz rings. For a positive integer n, let $\mathbb{Z}/n\mathbb{Z}$ denotes ring of integers modulo n. Obviously, for each positive integer n, the ring $\mathbb{Z}/n\mathbb{Z}$ is not reduced but Armendariz.

In the sequel we consider the relationship between the class of α-semicommutative modules and the class of Armendariz modules.

Theorem 2.26. *Let M be a right R-module. Then the followings hold.*

(1) *If M is an α-Armendariz R-module, then M is α-semicommutative if and only if $M[x;\alpha]$ is semicommutative as an $R[x;\alpha]$-module.*

(2) *If M is an Armendariz R-module, then M is α-semicommutative if and only if $M[x]$ is α-semicommutative as an $R[x]$-module.*

Proof.

(1) Let M be an α-Armendariz R-module. Assume that $M[x;\alpha]$ is a semicommutative $R[x;\alpha]$-module. By Lemma 2.23, M is an α-semicommutative module. Conversely, suppose that M is α-semicommutative. Let $m(x)f(x) = 0$, where $m(x) \in M[x]$, $f(x) \in R[x]$. Since M is α-Armendariz and α-semicommutative, we have $m_i a_j = 0$ and so $m_i R\alpha^n(a_j) = 0$, for $n = 1, 2, 3, \ldots$. For any positive integer t, $m_0 a\alpha^t(a_0) = 0$, $m_1\alpha(a)\alpha^{t+1}(a_1) = 0, \ldots, m_i\alpha(a)\alpha^{t+i}(a_i) = 0, \ldots$ from which we have

$$\begin{aligned} &m(x)ax^t f(x) \\ &\quad= \big(m_0 + m_1x + \cdots + m_nx^n\big)ax^t\big(a_0 + a_1x + \cdots + a_mx^m\big) \\ &\quad= \big(m_0ax^t + m_1\alpha(a)x^{t+1} + \cdots + m_n\alpha^n(a)x^{t+n}\big)\big(a_0 + \cdots + a_mx^m\big) \\ &\quad= m_0a\alpha^t(a_0)x^t + m_0a\alpha^t(a_1)x^{t+1} + \cdots + m_1\alpha(a)\alpha^{t+1}(a_0)x^{t+1} + \\ &\qquad\qquad m_1\alpha(a)\alpha^{t+1}(a_1)x^{t+2} + \cdots \\ &\quad= 0. \end{aligned}$$

It follows that $m(x)R[x;\alpha]f(x)=0$ and so $M[x;\alpha]$ is semicommutative.

(2) Let M be an Armendariz module. Suppose that $M[x]$ is α-semicommutative and $ma=0$ where $m\in M$, $a\in R$. We have $mR[x]\alpha(a)=0$ and so $mR\alpha(a)=0$. Hence M is α-semicommutative. Conversely, assume that M is α-semicommutative. Let $m(x)f(x)=0$ where $m(x)\in M[x]$, $f(x)\in R[x]$. We have $m_ia_j=0$ since M is Armendariz. By assumption, $m_iR\alpha(a_j)=0$. This implies that $m_iRx^t\alpha(a_j)=0$, in particular $m_iR[x]\alpha(a_j)=0$. Hence $m(x)R[x]\alpha\big(f(x)\big)=0$. Therefore $M[x]$ is α-semicommutative. ☑

Corollary 2.27. *[11, Corollary 3.2] Let M be an Armendariz right R-module. Then M is a semicommutative right R-module if and only if $M[x]$ is a semicommutative right $R[x]$-module.*

According to Hong, Kim and Kwak [4], R is called *α-skew Armendariz* if $p(x)q(x)=0$ where $p(x)=\sum_{i=0}^{m}a_ix^i$ and $q(x)=\sum_{j=0}^{n}b_jx^j\in R[x;\alpha]$ implies $a_i\alpha^i(b_j)=0$ for all $0\le i\le m$ and $0\le j\le n$. In [11], a module M is said to be *α-skew Armendariz* for any $m(x)=\sum_{i=0}^{n}m_ix^i\in M[x;\alpha]$ and $f(x)=\sum_{j=0}^{s}a_jx^j\in R[x;\alpha]$, $m(x)f(x)=0$ implies $m_i\alpha^i(a_j)=0$ for all i and j. Hence M is an Armendariz module if and only if it is $\mathbf{1}$-Armendariz if and only if it is $\mathbf{1}$-skew Armendariz. A ring R is skew-Armendariz if and only if the right R-module R_R is a skew-Armendariz module.

Theorem 2.28. *Let M be an α-semicommutative and rigid module. Then M is α-skew Armendariz. In particular, if M is semicommutative and rigid, then M is Armendariz.*

Proof. Let M be an α-semicommutative and rigid module. By Lemma 2.22, if $m(x)f(x)=0$, then $m_i\alpha^i(a_j)=0$ for all i and j, where $m(x)=\sum_{i=0}^{n}m_ix^i\in M[x;\alpha]$ and $f(x)=\sum_{j=0}^{s}a_jx^j\in R[x;\alpha]$. Hence M is α-skew Armendariz. In particular, if $\alpha=\mathbf{1}$, then M is Armendariz. ☑

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