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On Weak Solvability of Boundary Value Problems for Elliptic Systems

Sobre la solubilidad débil de problemas con valores en la frontera para sistemas elípticos

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Abstract. This paper concerns with existence and uniqueness of a weak solution for elliptic systems of partial differential equations with mixed boundary conditions. The proof is based on establishing the coerciveness of bilinear forms, related with the system of equations, which depend on first-order derivatives of vector functions in $\mathbb{R}^n$. The condition of coerciveness relates to Korn's type inequalities. The result is illustrated by an example of boundary value problems for a class of elliptic equations including the equations of linear elasticity.

Key words and phrases. Weak solvability, Boundary value problems, Elliptic equations, Korn's type inequality.

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Resumen. Este artículo trata sobre la existencia y unicidad de una solución débil para sistemas elípticos de ecuaciones diferenciales parciales con condiciones de frontera mixtas. La demostración se basa en la determinación de la coercividad de formas bilineales, relacionadas con el sistema de ecuaciones, las cuales dependen de las derivadas de primer orden de funciones vectoriales en $\mathbb{R}^n$. La condición de coercividad se relaciona con desigualdades tipo Korn. El resultado se ilustra mediante un ejemplo de problemas con valores en la frontera para una clase de ecuaciones elípticas, incluyendo las ecuaciones de elasticidad lineal.

Palabras y frases clave. Solubilidad débil, problemas con valores en la frontera, ecuaciones elípticas, desigualdad tipo Korn.

1. Introduction

Systems of elliptic partial differential equations arise frequently in problems of continuum mechanics. To formulate boundary value problems (BVPs), we should supply the equations with some boundary conditions. We will consider the linear partial differential equations derived from an energy type functional. We will get solutions to a BVP minimizing the energy type functional. Such a solution will be called weak. As we expect to apply the results to physical problems we use the terminology like "energy", "displacements", etc.

In general case solution of a BVP will be reduced to the minimization problem for a functional of the form $E(u) = \frac{1}{2}B(u,u) - f(u)$ in a Hilbert space H, where B is a symmetric bilinear form and f is a linear functional in H. It is well know that

Theorem 1.1 ([15, 16])**.** *Let B be a continuous symmetric bilinear form in a Hilbert space H. If f is a continuous linear functional and $B(u,u) \geq C\|u\|^2$ for every $u \in H$ and some constant $C > 0$, then E attains a unique minimum in H. Furthermore, u_0 is the minimum point if and only if it is a unique solution to equation*

$$B(u_0, v) = f(v) \tag{1}$$

for every $v \in H$.

This simple result allows us to prove existence–uniqueness theorems for mechanical problems; it motivates the following definition.

Definition 1.2. A bilinear form B in a Hilbert space is coercive if $B(u,u) \geq C\|u\|^2$ for every $u \in H$ and some constant C.

In the proof of existence and uniqueness of weak solutions of the problems under consideration, the most troublesome point is to establish the coerciveness of B. In linear elasticity such coerciveness is also called Korn's inequality which was first proved in [13, 14]. Subsequent generalizations of Korn's inequality can be found in [9, 5, 17, 11, 6, 1]. One of the most important latest papers on Korn's inequality is due to V. A. Kondrat'ev and O. A. Oleynik [12], where it is established in a general form.

Similar problems frequently arise in equilibrium or stationary linear boundary value physical problems [3, 4, 7, 8, 10]. We extend some known results for linear elasticity to a general case and consider a general coerciveness problem of B, construction of abstract energy spaces related to B as well as the weak setup of corresponding boundary value problems and their solvability.

Assume that Ω is a connected bounded open set in $\mathbb{R}^n$ with piecewise C^1 boundary $\partial\Omega$. Let B be a positive bilinear form, that is, if $B\big(\boldsymbol{u},\boldsymbol{u}\big) = \mathbf{0}$ for a smooth vector function $\boldsymbol{u} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$, such that $\boldsymbol{u}|_\Gamma = \mathbf{0}$ for some open set $\Gamma \subset \partial\Omega$, it follows that $\boldsymbol{u} = \mathbf{0}$. Later we will show when this condition is valid. The linear space of smooth functions with $\boldsymbol{u}|_\Gamma = \boldsymbol{0}$ constitutes an inner product space if we take $B(\boldsymbol{u},\boldsymbol{v})$ as an inner product. Completion of this space with respect to the norm induced by the inner product is an energy space.

The bilinear form used for the inner product is

$$B(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \sum_{i,j=1}^{m} a^{ij}(x) L_i(\nabla \boldsymbol{u}, \boldsymbol{u}, x) L_j(\nabla \boldsymbol{v}, \boldsymbol{v}, x)\, dx \tag{2}$$

where $a^{ij} \in L^\infty(\Omega)$ are components of a symmetric positive definite matrix and the linear forms L_i are

$$L_i(\nabla \boldsymbol{u}, \boldsymbol{u}, x) = \sum_{k,l=1}^{n} b_i^{kl}(x)\partial_k u_l + \sum_{k=1}^{n} c_i^k(x) u_k.$$

Let $\boldsymbol{f} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ and $\boldsymbol{g} : \partial\Omega - \Gamma \to \mathbb{R}^n$ be vector functions, these define the energy functional

$$E(\boldsymbol{u}) = \frac{1}{2} B(\boldsymbol{u},\boldsymbol{u}) - \int_\Omega \boldsymbol{f}\boldsymbol{\cdot}\boldsymbol{u}\, dx - \int_{\partial\Omega-\Gamma} \boldsymbol{g}\boldsymbol{\cdot}\boldsymbol{u}\, dS. \tag{3}$$

By analogy with elasticity equations, we present B in the form

$$\begin{aligned} &B(\boldsymbol{u},\boldsymbol{v}) = \\ &\int_\Omega \sum_t \left[-\sum_{k,s} \partial_s \left(\sum_l d^{klst} \partial_k u_l + q^{kst} u_k \right) + \sum_{k,l} q^{tkl} \partial_k u_l + \sum_k p^{kt} u_k \right] v_t\, dx \, + \\ &\qquad\qquad \int_{\partial\Omega-\Gamma} \sum_t \left[\sum_{k,l,s} d^{klst} \partial_k u_l n_s + \sum_{k,s} q^{kst} u_k n_s \right] v_t\, dS, \end{aligned}$$

where dS is the surface area element, $\boldsymbol{n}$ the normal vector to the surface and the coefficients are

$$d^{klst} = \sum_{i,j} a^{ij} b_i^{kl} b_j^{st}, \qquad q^{kst} = \sum_{i,j} a^{ij} c_i^k b_j^{st}, \qquad p^{kt} = \sum_{i,j} a^{ij} c_i^k c_j^t.$$

If the functions involved in E are sufficiently smooth and satisfy the boundary condition, functional E can be obtained from (2) integrating by parts. Using standard methods of calculus of variations, assuming existence of a smooth minimizer $\boldsymbol{u}$ of E and minimizing E over the set of smooth functions equal to zero on Γ, we get the corresponding system of partial differential equations

$$-\sum_{k,s} \partial_s \left(\sum_l d^{klst} \partial_k u_l + q^{kst} u_k \right) + \sum_{k,l} q^{tkl} \partial_k u_l + \sum_k p^{kt} u_k = f_t \tag{4}$$

for $t = 1, \dots, n$. Moreover, on $\partial\Omega - \Gamma$ we obtain the natural condition

$$\sum_{k,s} \left(\sum_{l} d^{klst} \partial_k u_l + q^{kst} u_k \right) n_s = g_t \tag{5}$$

for $t = 1, \dots, n$, that is analogous to Neumann's condition for the Laplace equation, whereas on Γ we have Dirichlet condition

$$\boldsymbol{u}|_\Gamma = 0. \tag{6}$$

Definition 1.3. A vector function $\boldsymbol{u}$ is a weak solution of the system in (4) with boundary conditions (5) and (6) if it is a minimum point of the energy functional $E(\boldsymbol{u})$ in (3).

We will present some sufficient conditions for coerciveness and continuity of B in the Hilbert space $\boldsymbol{H}$ of vector functions in $\left[W^{1,2}(\Omega)\right]^n$ with zero trace value in $\Gamma \subset \partial\Omega$. In this way we will show that the energy norm $B(\boldsymbol{u}, \boldsymbol{u})^{\frac{1}{2}}$ is an equivalent norm in $\boldsymbol{H}$, which in turn insure existence and uniqueness of a weak solution of the boundary value problem (4)–(6) in the space $\boldsymbol{H}$. We apply this result to a particular type of bilinear forms, generalizing a result from the theory of shallow shells [18] and consider an example of boundary value problems for a class of elliptic equations including the equations of linear elasticity.

Notation. The norm of a vector field $\boldsymbol{u}$ in $\left[L^q(\Omega)\right]^n = L^q(\Omega) \times \cdots \times L^q(\Omega)$ and the $L^2(\Omega)$ norm of $\nabla \boldsymbol{u}$ are, respectively,

$$\|\boldsymbol{u}\|_q = \left\{ \int_\Omega \sum_{i=1}^n |u_i|^q \right\}^{\frac{1}{q}} \qquad \text{and} \qquad \left\|\nabla \boldsymbol{u}\right\|_2 = \left\{ \int_\Omega \sum_{i,j=1}^n \left|\partial_i u_j\right|^2 \, dx \right\}^{\frac{1}{2}}.$$

The norm of $\left[W^{1,2}(\Omega)\right]^n = W^{1,2}(\Omega) \times \cdots \times W^{1,2}(\Omega)$ is

$$\|\boldsymbol{u}\|_{1,2} = \left\{ \|\boldsymbol{u}\|_2^2 + \|\nabla \boldsymbol{u}\|_2^2 \right\}^{\frac{1}{2}}.$$

2. Equivalence of Energy and Sobolev's Norms

Let us rewrite the symmetric bilinear functional

$$B(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \sum_{i,j=1}^m a^{ij}(x) L_i(\nabla \boldsymbol{u}, \boldsymbol{u}, x) L_j(\nabla \boldsymbol{v}, \boldsymbol{v}, x) \, dx \tag{7}$$

where $\left(a^{ij}\right)$ is a symmetric positive definite matrix almost everywhere (a.e.) and the linear forms L_i are

$$L_i(\nabla \boldsymbol{u}, \boldsymbol{u}, x) = \sum_{k,l=1}^n b_i^{kl}(x) \partial_k u_l + \sum_{k=1}^n c_i^k(x) u_k.$$

We can also write

$$L_i\big(\nabla \boldsymbol{u}, \boldsymbol{u}, x\big) = M_i\big(\nabla \boldsymbol{u}, x\big) + N_i\big(\boldsymbol{u}, x\big),$$

where M_i and N_i are linear forms. In what follows, we assume that

$$a^{ij}, b_i^{kl} \in L^\infty(\Omega) \tag{8}$$

$$c_i^k \in L^q(\Omega) \quad \text{for} \quad n < q \leq \infty. \tag{9}$$

By the positive definiteness of $\big(a^{ij}\big)$, there exists a positive bounded a.e. function h such that

$$B\big(\boldsymbol{u}, \boldsymbol{u}\big) \geq \int_\Omega \sum_{i=1}^m h(x)\big(L_i\big(\nabla \boldsymbol{u}, \boldsymbol{u}, x\big)\big)^2 \, dx.$$

From this, it is easy to prove that B is positive definite if and only if $L_i\big(\nabla \boldsymbol{u}, \boldsymbol{u}, x\big) = 0$ a.e. for every i implies that $\boldsymbol{u} = \boldsymbol{0}$, a.e.

We will use the following classical properties of Sobolev spaces

Theorem 2.1 (The Embedding theorem [2]). *Suppose that Ω is a bounded open set satisfying the cone condition. Then*

(i) If $n = 2$, $W^{1,2}(\Omega)$ is continuously and compactly embedded in $L^q(\Omega)$, for $1 \leq q < \infty$; that is, for every $u \in W^{1,2}(\Omega)$, there exists a constant C such that $\|u\|_q \leq C\|u\|_{1,2}$ and if $u_n \rightharpoonup u$ in $W^{1,2}(\Omega)$, then $u_n \to 0$ in $L^q(\Omega)$.

(ii) If $n > 2$, $W^{1,2}(\Omega)$ is continuously embedded in $L^q(\Omega)$, for $1 \leq q \leq \frac{2n}{n-2}$, and compactly embedded in $L^q(\Omega)$, for $1 \leq q < \frac{2n}{n-2}$.

The continuous embedding is known as Sobolev embedding theorem, and the compact embedding as Rellich-Kondrachov theorem.

The goal of this section is to show that the terms M_i, which include only first order derivatives, determine the equivalence of the energy norm with the Sobolev's norm.

Theorem 2.2. *If B is the bilinear form given by the Equation (7) and*

$$\int_\Omega \sum_{i,j=1}^m a^{ij}(x) M_i\big(\nabla \boldsymbol{u}, x\big) M_j\big(\nabla \boldsymbol{u}, x\big) \, dx \geq C\big\|\nabla \boldsymbol{u}\big\|_2^2, \tag{10}$$

then the energy norm $B\big(\boldsymbol{u}, \boldsymbol{u}\big)^{\frac{1}{2}}$ is equivalent to the norm of $\boldsymbol{H}$.

Corollary 2.3. *Suppose that B satisfies the hypothesis of Theorem 2.2 and let $\boldsymbol{f} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ and $\boldsymbol{g} : \partial\Omega - \Gamma \to \mathbb{R}^n$ be vector functions. If*

$$\boldsymbol{f} \in \left[L^p(\Omega)\right]^n \quad \textit{and} \quad \boldsymbol{g} \in \left[L^{p^*}(\partial\Omega - \Gamma)\right]^n \quad \textit{for} \quad 1 < p,\ p^* \leq \infty, \quad \textit{if} \quad n = 2,$$
$$\boldsymbol{f} \in \left[L^{\frac{2n}{n+2}}(\Omega)\right]^n \quad \textit{and} \quad \boldsymbol{g} \in \left[L^{\frac{2(n-1)}{n}}(\partial\Omega - \Gamma)\right]^n, \qquad \textit{if} \quad n > 2.$$

Then for the corresponding boundary value problem (4)–(6) *there exists a unique weak solution in* $\boldsymbol{H}$.

Proof. It is a consequence of Theorem 1.1 and the continuity of the functionals $\int_\Omega \boldsymbol{f} \bullet \boldsymbol{v}\, dx$ and $\int_{\partial\Omega-\Gamma} \boldsymbol{g} \bullet \boldsymbol{v}\, dx$ in $\left[W^{1,2}(\Omega)\right]^n$. ☑

Inequality (10) is of Korn's type. The name comes from the classical Korn's inequality in elasticity, which depends strongly on the fact that the vector field is zero on some open subset of the boundary

$$\int_\Omega \sum_{i,j=1}^n (\partial_i v_j + \partial_j v_i)^2\, dx \geq C\|\nabla \boldsymbol{v}\|_2^2.$$

Before proving Theorem 2.2, we need the following lemma.

Lemma 2.4. *Suppose that* Ω *is a bounded open set satisfying the cone condition and* $\boldsymbol{Z} \subset \left[W^{1,2}(\Omega)\right]^n$ *is a closed subspace. Let* B *be a continuous positive definite bilinear form in* $\boldsymbol{Z}$. *If the conditions* $\boldsymbol{v}_k \rightharpoonup 0$ *in* $\left[W^{1,2}(\Omega)\right]^n$ *and* $B(\boldsymbol{v}_k, \boldsymbol{v}_k) \to 0$ *imply that* $\|\nabla \boldsymbol{v}_k\|_2 \to 0$, *then*

$$B(\boldsymbol{v}, \boldsymbol{v}) \geq C\|\boldsymbol{v}\|_{1,2}^2 \tag{11}$$

for some constant C.

Proof. Suppose contrarily that $B(\boldsymbol{v}, \boldsymbol{v}) \geq C\|\boldsymbol{v}\|_{1,2}^2$ is false for any C. Then we can find a sequence $\{\boldsymbol{v}_k\}$ such that $B(\boldsymbol{v}_k, \boldsymbol{v}_k) \to 0$ and $\|\boldsymbol{v}_k\|_{1,2} = 1$. As $\boldsymbol{Z}$ is a Hilbert space and $\{\boldsymbol{v}_k\}$ is bounded, we can assume that the sequence converges weakly to some vector $\boldsymbol{v}_0$. As $B(\bullet, \boldsymbol{v}_0)$ is a continuous functional, by positive definiteness of B,

$$0 \leq B(\boldsymbol{v}_k - \boldsymbol{v}_0, \boldsymbol{v}_k - \boldsymbol{v}_0) \leq B(\boldsymbol{v}_k, \boldsymbol{v}_k) - 2B(\boldsymbol{v}_k, \boldsymbol{v}_0) + B(\boldsymbol{v}_0, \boldsymbol{v}_0).$$

Taking the limit when $k \to \infty$ we get

$$0 \leq -B(\boldsymbol{v}_0, \boldsymbol{v}_0).$$

So, $B(\boldsymbol{v}_0, \boldsymbol{v}_0) = 0$, which implies that $\boldsymbol{v}_0 = \boldsymbol{0}$. Thus $\boldsymbol{v}_k \rightharpoonup 0$ in $\left[W^{1,2}(\Omega)\right]^n$. As $B(\boldsymbol{v}_k, \boldsymbol{v}_k) \to 0$. by the assumptions we conclude that $\left\|\nabla \boldsymbol{v}_k\right\|_2 \to 0$.

As $\boldsymbol{v}_k \rightharpoonup 0$ in $\left[W^{1,2}(\Omega)\right]^n$, by Rellich–Kondrachov's theorem we see that $\boldsymbol{v}_k \to 0$ in $\left[L^2(\Omega)\right]^n$. Thus $\boldsymbol{v}_k \to 0$ in $\left[W^{1,2}(\Omega)\right]^n$, which contradicts the equality $\left\|\boldsymbol{v}_k\right\|_{1,2} = 1$. ☑

Proof of Theorem 2.2. To verify the continuity of the bilinear form B, by the symmetry of B it suffices to show that $\left|B\big(\boldsymbol{u},\boldsymbol{u}\big)\right| \le C\|\boldsymbol{u}\|_{1,2}^2$. Choosing p according to (9) and q such that $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$, for the components of the bilinear form we have

$$\begin{aligned}
\left|\int_\Omega a^{ij}b_i^{kl}b_j^{st}\partial_k u_l \partial_s u_t\,dx\right| &\le \frac{1}{2}\left\|a^{ij}b_i^{kl}b_j^{st}\right\|_\infty\big(\|\partial_k u_l\|_2^2+\|\partial_s u_t\|_2^2\big)\\
&\le C_1\left\|\boldsymbol{u}\right\|_{1,2}^2, &(12)\\
\left|\int_\Omega a^{ij}b_i^{kl}c_j^{s}\partial_k u_l u_s\,dx\right| &\le \left\|a^{ij}b_i^{kl}\right\|_\infty\left\|c_j^s\right\|_p\|\partial_k u_l\|_2\|u_s\|_q\\
&\le C_2\left\|\boldsymbol{u}\right\|_{1,2}^2, &(13)\\
\left|\int_\Omega a^{ij}c_i^{k}c_j^{s}u_k u_s\,dx\right| &\le \frac{1}{2}\left\|a^{ij}\right\|_\infty\Big(\left\|c_i^k\right\|_p^2\|u_k\|_q^2+\left\|c_j^s\right\|_p^2\|u_s\|_q^2\Big)\\
&\le C_3\left\|\boldsymbol{u}\right\|_{1,2}^2. &(14)
\end{aligned}$$

So we conclude that B is continuous.

By Lemma 2.4, now we only need to prove that $\boldsymbol{u}_k \rightharpoonup 0$ in $\left[W^{1,2}(\Omega)\right]^n$ and $B\big(\boldsymbol{u}_k,\boldsymbol{u}_k\big)\to 0$ together imply that $\left\|\nabla\boldsymbol{u}_k\right\|_2\to 0$. We write the bilinear form as follows:

$$\begin{aligned}
B\big(\boldsymbol{u}_k,\boldsymbol{u}_k\big) = \int_\Omega\sum_{i,j=1}^m a^{ij}(x)M_i\big(\nabla\boldsymbol{u}_k,x\big)M_j\big(\nabla\boldsymbol{u}_k,x\big)\,dx\,+\\
2\int_\Omega\sum_{i,j=1}^m a^{ij}(x)M_i\big(\nabla\boldsymbol{u}_k,x\big)N_j\big(\boldsymbol{u}_k,x\big)\,dx\,+\\
\int_\Omega\sum_{i,j=1}^m a^{ij}(x)N_i\big(\boldsymbol{u}_k,x\big)N_j\big(\boldsymbol{u}_k,x\big)\,dx. \qquad (15)
\end{aligned}$$

By Rellich-Kondrachov's theorem we know that $\left\|\boldsymbol{u}_k\right\|_q\to 0$. The first Inequality in (14) shows that the third term on the right-hand side of the Inequality (15) tends to zero. In a similar fashion, by the first inequality in (13) and the boundedness of $\|\partial_i u_{k,j}\|_2$, the second term also tends to zero. Thus, by Equation (10), we get $\left\|\nabla\boldsymbol{u}_k\right\|_2\to 0$ as $k\to\infty$, which concludes the proof. ☑

When we try to prove the coerciveness of B, the uniform positive definiteness of $\big(a^{ij}\big)$ allows us to simplify the proof and to extend theorems on equivalence between energy norms and Sobolev's norms.

Definition 2.5. The terms $\big(a^{ij}\big)$ are uniformly positive definite if

$$\sum_{i,j=1}^m a^{ij}(x)\xi_i\xi_j \ge \theta\sum_{i=1}^m \xi_i^2$$

for every $x \in \Omega$ and some constant $\theta > 0$.

This stronger requirement implies that

$$B\big(\boldsymbol{u},\boldsymbol{u}\big) \geq \theta \int_\Omega \sum_{i=1}^m \big(L_i\big(\nabla \boldsymbol{u},\boldsymbol{u},x\big)\big)^2 \, dx.$$

In general, it is easier to prove the coerciveness of the form in the right hand side, obtaining thus the coerciveness of the original bilinear form. Moreover, once it is proved that

$$\widehat{B}\big(\boldsymbol{u},\boldsymbol{v}\big) = \int_\Omega \sum_{i=1}^m L_i\big(\nabla \boldsymbol{u},\boldsymbol{u},x\big) L_i\big(\nabla \boldsymbol{v},\boldsymbol{v},x\big)\, dx \tag{16}$$

is coercive, we can extend this result to a wider class of bilinear forms.

Theorem 2.6. *Suppose that* $\big(a^{ij}(x)\big)$ *is uniformly positive definite and*

$$\begin{aligned} \boldsymbol{T}(x) &: \mathbb{R}^m \to \mathbb{R}^m \\ (v_i) &\mapsto \left(\sum_{t=1}^m t_{ij}(x) v_j\right) \end{aligned} \tag{17}$$

is a non-singular linear operator at each point, such that $\big\|\boldsymbol{T}(x)^{-1}\big\|$ *is bounded. The bilinear form* B *defined by*

$$\int_\Omega \sum_{i,j=1}^m a^{ij} \left(\sum_{t=1}^m t_{it}(x) L_t\big(\nabla \boldsymbol{u},\boldsymbol{u},x\big)\right)\left(\sum_{t=1}^m t_{jt}(x) L_t\big(\nabla \boldsymbol{v},\boldsymbol{v},x\big)\right) dx \tag{18}$$

is coercive if $\widehat{B}$ *in* (16) *is coercive.*

Proof. By uniform coerciveness of $\big(a^{ij}\big)$, the theorem follows from the inequality

$$\sum_{i=1}^m \left(\sum_{j=1}^m t_{ij}(x) L_j\big(\nabla \boldsymbol{u},\boldsymbol{u},x\big)\right)^2 \geq C \sum_{i=1}^m \big(L_i\big(\nabla \boldsymbol{u},\boldsymbol{u},x\big)\big)^2,$$

where $C > 0$ is a constant, after integrating over Ω. But this inequality is an easy consequence of

$$\Big|\big[\boldsymbol{T}(x)\big](v)\Big| \geq \frac{1}{\big\|\boldsymbol{T}(x)^{-1}\big\|}\,|v| \geq \frac{1}{C_1}\,|v|$$

where $|\bullet|$ is the euclidean norm in $\mathbb{R}^m$ and $\big\|\boldsymbol{T}(x)^{-1}\big\| \leq C_1 < \infty$. ☑

3. Generalization of a Bilinear Form in Elasticity

The following bilinear form is an extension of the body strain energy in theory of shallow shells,

$$B\big(\boldsymbol{u},\boldsymbol{v}\big)=\int_\Omega \sum_{i,j,k,l=1}^{n} a^{ijkl}(x)L_{ij}\big(\nabla\boldsymbol{u},\boldsymbol{u},x\big)L_{kl}\big(\nabla\boldsymbol{v},\boldsymbol{v},x\big)\,dx$$

where $a^{ijkl}=a^{klij}\in L^\infty(\Omega)$ are uniformly positive definite, that is, for some $\theta>0$ and every $x\in\Omega$ we have

$$\sum_{i,j,k,l=1}^{n} a^{ijkl}(x)\xi_{ij}\xi_{kl}\geq\theta\sum_{i,j=1}^{n}\xi_{ij}^2.$$

The linear components are

$$L_{ij}(\boldsymbol{v})=\partial_i v_j+\partial_j v_i+c_{ij}^k v_k$$

where $c_{ij}^k\in L^\infty(\Omega)$.

By the uniform positive definiteness of $\big(a^{ijkl}\big)$, instead of B, we prove the coerciveness of

$$\widehat{B}(\boldsymbol{u},\boldsymbol{v})=\int_\Omega\sum_{i,j=1}^{n}L_{ij}\big(\nabla\boldsymbol{u},\boldsymbol{u},x\big)L_{ij}(\nabla\boldsymbol{v},\boldsymbol{v},x)\,dx.$$

As noted in the preceding section, by means of Theorem 2.6, we can extend equivalence of the energy norm $B\big(\boldsymbol{u},\boldsymbol{u}\big)^{\frac12}$ to a greater class of energy norms. Linear forms L_{ij} can be considered as the components in the canonical basis of operator

$$\boldsymbol{L}(\boldsymbol{v})=\nabla\boldsymbol{v}+\nabla\boldsymbol{v}^T+\boldsymbol{C}\bullet\boldsymbol{v},$$

where the components of $\boldsymbol{C}\bullet\boldsymbol{v}$ are $\big(\sum_k c_{ij}^k v_k\big)$. So we have the identity

$$\widehat{B}\big(\boldsymbol{u},\boldsymbol{v}\big)=\int_\Omega \operatorname{tr}\Big(\boldsymbol{L}(\boldsymbol{u})\boldsymbol{L}(\boldsymbol{v})^T\Big)\,dx. \tag{19}$$

An additional advantage of $\widehat{B}$ is its invariance under orthogonal transformations. To see this, suppose $\{e_i\}$ is an orthonormal basis for $\mathbb{R}^n$ and let $e_{i'}=\sum_i q_{i'}^i e_i$ define a new orthonormal basis, whose inverse transformation is $e_i=\sum_i q_i^{i'}e_{i'}$, where $\sum_{j'}q_i^{j'}q_{j'}^k=\delta_i^k$. As neither $\nabla\boldsymbol{v}$ nor $\boldsymbol{C}\bullet\boldsymbol{v}$ depend on a particular basis and the trace of a matrix is invariant under orthogonal transformations, we have from Equation (19) that

$$\int_\Omega \sum_{i,j=1}^{n} \left(\partial_i v_j + \partial_j v_i + \sum_{k=1}^{n} c_{ij}^k v_k \right)^2 dx = \int_{\Omega'} \sum_{i',j'=1}^{n} \left(\partial_{i'} v_{j'} + \partial_{j'} v_{i'} + \sum_{k'=1}^{n} c_{i'j'}^{k'} v_{k'} \right)^2 dx'$$

where $c_{i'j'}^{k'} = \sum_{i,j,k} c_{ij}^k q_k^{k'} q_{i'}^i q_{j'}^j$ and Ω' is the domain in the new coordinates. Since the norm is an invariant, it can be seen that

$$\|c_{i'j'}^{k'}\|_\infty^2 \leq \sum_{i,j,k=1}^{n} \|c_{ij}^k\|_\infty^2 = \|\boldsymbol{C}\|_\infty^2.$$

As equation (10) holds by the classical Korn's inequality, it remains to prove that $\widehat{B}$ is positive definite.

Theorem 3.1. *If $\widehat{B}(\boldsymbol{u}, \boldsymbol{u}) = 0$, then $\boldsymbol{u} = \mathbf{0}$ a.e.*

***Proof*.** If $\widehat{B}(\boldsymbol{u}, \boldsymbol{u}) = 0$, then for any basis of $\mathbb{R}^n$

$$2\partial_{i'} u_{i'} = -\sum_{k'=1}^{n} c_{i'i'}^{k'} u_{k'}.$$

Let us draw, rotating if it is necessary, a hypercube with side of length L such that Γ passes through adjacent sides of the hypercube (Figure 1(a)). The set of points in Ω enclosed by Γ and the hypercube is called V. The faces of the hypercube in Ω' are the planes $x_{i'} = a_{i'}$.

Let $n_{i'}$ be the components of the normal vector to the surface. The function $(x_{i'} - a_{i'}) u_{i'}^2 n_{i'}$ is zero on ∂V. Thus, using Gauss-Green formula, we get

$$\int_V \partial_{i'} \Big((x_{i'} - a_{i'}) u_{i'}^2 \Big)\, dx = \int_V u_{i'}^2 + 2(x_{i'} - a_{i'}) u_{i'} \partial_{i'} u_{i'}\, dx' = 0.$$

It follows

$$\begin{aligned}
\int_V u_{i'}^2\, dx' &\leq 2 \int_V \big| (x_{i'} - a_{i'}) u_{i'} \partial_{i'} u_{i'} \big|\, dx' \\
&\leq \int_V \left| (x_{i'} - a_{i'}) u_{i'} \sum_{k'=1}^{n} c_{i'i'}^{k'} u_{k'} \right| dx' \\
&\leq \frac{L}{2} \int_V \sum_{k'=1}^{n} \big| c_{i'i'}^{k'} \big| \big(u_{i'}^2 + u_{k'}^2 \big)\, dx' \\
&= \frac{L}{2} \int_V \left(\big| c_{i'i'}^{i'} \big| + \sum_{k'=1}^{n} \big| c_{i'i'}^{k'} \big| \right) u_{i'}^2 + \sum_{k' \neq i'} \big| c_{i'i'}^{k'} \big| u_{k'}^2\, dx'
\end{aligned}$$

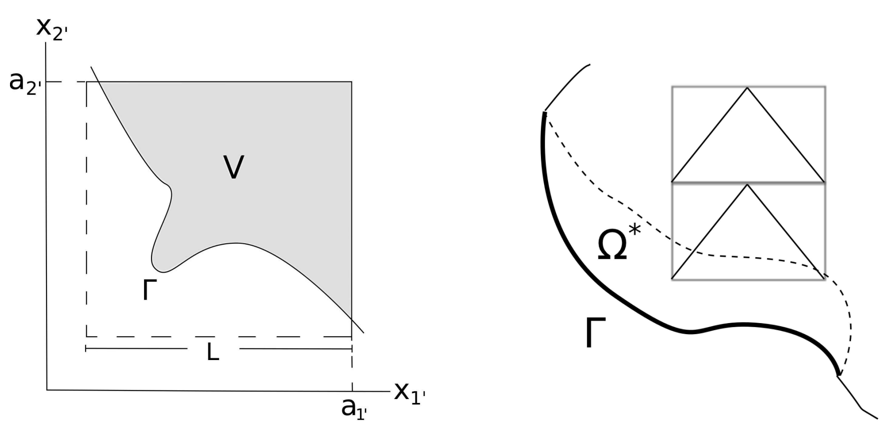
$$
\begin{aligned}
&\leq L \sum_{k'=1}^{n} \left\| c_{i'i'}^{k'} \right\|_{\infty} \int_V \|\boldsymbol{u}\|^2 \, dx' \\
&\leq nL \left\| \boldsymbol{C} \right\|_{\infty} \int_V \|\boldsymbol{u}\|^2 \, dx'.
\end{aligned}
$$

Taking the sum over all the components of $\boldsymbol{u}$ we have

$$\int_V \|\boldsymbol{u}\|^2 \, dx' \leq n^2 L \left\| \boldsymbol{C} \right\|_{\infty} \int_V \|\boldsymbol{u}\|^2 \, dx'. \tag{20}$$

Choosing L small enough so that $n^2 L \left\| \boldsymbol{C} \right\|_{\infty} < 1$, we get $\boldsymbol{u} = \boldsymbol{0}$ a.e. in V. This inequality makes sense for $\boldsymbol{u} \in \left[W^{1,2}(\Omega) \right]^n$. It is important to note that the length L does not depend on the location of the hyperrectangles in Ω.

As for every point in Γ there exists a neighborhood where $\boldsymbol{u} = \boldsymbol{0}$ a.e., then in a neighborhood Ω^* of Γ we have $\boldsymbol{u} = \boldsymbol{0}$ a.e.. From this neighborhood, using hyperrectangles as in Figure 1(b) we can extend the equality $\boldsymbol{u} = \boldsymbol{0}$ a.e.. The set Ω can be covered by a net of hyperrectangles stemming from Ω^*, covering a subset Ω_1 (Figure 2). Next, we use smaller hyperrectangles covering a greater set $\Omega_2 \subset \Omega$ and so on, using at most countable many hyperrectangles, obtaining $\boldsymbol{u} = \boldsymbol{0}$ a.e. in Ω. ☑



(a) Area of integration.

(b) Extension by rectangles.

FIGURE 1

The system of equations related to B is

$$-\sum_{i,j} \partial_l \left(d^{ijkl} \left(e_{ij} + \sum_s c_{ij}^s u_s \right) \right) + \sum_{i,j} q^{ijk} \left(e_{ij} + \sum_s c_{ij}^s u_s \right) = f_k \tag{21}$$

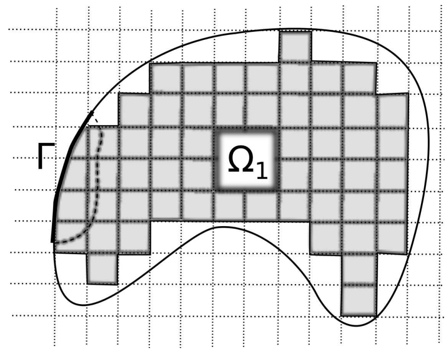


FIGURE 2. The function $\boldsymbol{u}$ is zero in the region Ω_1, depicted by gray rectangles.

where $d^{ijkl} = a^{ijkl} + a^{ijlk}$, $q^{ijk} = \sum_{s,t} a^{ijst} c^k_{st}$ and $e_{ij} = \partial_i u_j + \partial_j u_i$. Natural conditions are

$$\sum_{i,j,l} d^{ijkl}\left(e_{ij} + \sum_s c^s_{ij} u_s\right) n_l = g_k. \tag{22}$$

So we have established the following theorem.

Theorem 3.2. *Let $\boldsymbol{f} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ and $\boldsymbol{g} : \partial\Omega - \Gamma \to \mathbb{R}^n$ be vector functions such that*

$$\begin{aligned} &\boldsymbol{f} \in \left[L^p(\Omega)\right]^n \quad \textit{and} \quad \boldsymbol{g} \in \left[L^{p^*}(\partial\Omega - \Gamma)\right]^n \quad \textit{for} \quad 1 < p, p^* \leq \infty, \quad \textit{if} \quad n = 2,\\ &\boldsymbol{f} \in \left[L^{\frac{2n}{n+2}}(\Omega)\right]^n \quad \textit{and} \quad \boldsymbol{g} \in \left[L^{\frac{2(n-1)}{n}}(\partial\Omega - \Gamma)\right]^n, \qquad \textit{if} \quad n > 2. \end{aligned}$$

Then for the system of partial differential equations (21) *with boundary conditions* (6) *and* (22) *exists a unique solution in $\boldsymbol{H}$.*

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