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The Stekloff Problem for Rotationally Invariant Metrics on the Ball

El problema de Stekloff para métricas rotacionalmente invariantes en la bola

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Abstract. Let (B_r, g) be a ball of radius $r > 0$ in $\mathbb{R}^n$ $(n \geq 2)$ endowed with a rotationally invariant metric $ds^2 + f^2(s)dw^2$, where dw^2 represents the standard metric on S^{n-1}, the $(n-1)$–dimensional unit sphere. Assume that B_r has non–negative sectional curvature. In this paper we prove that if $h(r) > 0$ is the mean curvature on ∂B_r and ν_1 is the first eigenvalue of the Stekloff problem, then $\nu_1 \geq h(r)$. Equality $\big(\nu_1 = h(r)\big)$ holds only for the standard metric of $\mathbb{R}^n$.

Key words and phrases. Stekloff eigenvalue, Rotationally invariant metric, Non-negative sectional curvature.

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Resumen. Sea (B_r, g) una bola de radio $r > 0$ en $\mathbb{R}^n$ $(n \geq 2)$ dotada con una métrica g rotacionalmente invariante $ds^2 + f^2(s)dw^2$, donde dw^2 representa la métrica estándar sobre S^{n-1}, la esfera unitaria $(n-1)$–dimensional. Asumamos que B_r tiene curvatura seccional no negativa. En este artículo demostramos que si $h(r) > 0$ es la curvatura media sobre ∂B_r y ν_1 es el primer valor propio del problema de Stekloff, entonces $\nu_1 \geq h(r)$. La igualdad $\big(\nu_1 = h(r)\big)$ se tiene sólo si g es la métrica estándar de $\mathbb{R}^n$.

Palabras y frases clave. Valor propio de Stekloff, métrica rotacionalmente invariante, curvatura seccional no negativa.

1. Introduction

Let (M^n, g) be a compact Riemannian manifold with boundary. The Stekloff problem is the following: find a solution for the equation

$$\begin{aligned} &\Delta\varphi = 0 \quad \text{in} \quad M, \\ &\frac{\partial\varphi}{\partial\eta} = \nu\varphi \quad \text{on} \quad \partial M, \end{aligned} \tag{1}$$

where ν is a real number. Problem (1) for bounded domains in the plane was introduced by Stekloff in 1902 (see [7]). His motivation came from physics. The function φ represents a steady state temperature on M such that the flux on the boundary is proportional to the temperature. Problem (1) is also important in conductivity and harmonic analysis as it was initially studied by Calderón in [1]. This is because the set of eigenvalues for the Stekloff problem is the same as the set of eigenvalues of the well-known Dirichlet-Neumann map. This map associates to each function u defined on the boundary ∂M, the normal derivative of the harmonic function on M with boundary data u. The Stekloff problem is also important in conformal geometry in the problem of conformal deformation of a Riemannian metric on manifolds with boundary. The set of eigenvalues consists of an infinite sequence $0 < \nu_1 \le \nu_2 \le \nu_3 \le \cdots$ such that $\nu_i \to \infty$. The first non–zero eigenvalue ν_1 has the following variational characterization,

$$\nu_1 = \min_{\varphi}\left\{ \frac{\int_M |\nabla\varphi|^2\, dv}{\int_{\partial M}\varphi^2\, d\sigma} : \varphi \in C^\infty\big(\overline{M}\big), \int_{\partial M}\varphi\, d\sigma = 0 \right\}. \tag{2}$$

For convex domains in the plane, Payne (see [6]) showed that $\nu_1 \ge k_o$, where k_o is the minimum value of the curvature on the boundary of the domain. Escobar (see [2]) generalized Payne's Theorem (see [6]) to manifolds 2-dimensionals with non–negative Gaussian curvature. In this case Escobar showed that $\nu_1 \ge k_0$, where $k_g \ge k_0$ and k_g represents the geodesic curvature of the boundary. In higher dimensions, $n \ge 3$, for non–negative Ricci curvature manifolds, Escobar shows that $\nu_1 > \frac{1}{2}k_0$, where k_0 is a lower bound for any eigenvalue of the second fundamental form of the boundary. Escobar (see [3]) established the following conjecture.

Conjecture 1.1. Let (M^n, g) be a compact Riemannian with boundary and dimension $n \ge 3$. Assume that $\mathrm{Ric}(g) \ge 0$ and that the second fundamental form π satisfies $\pi \ge k_0 I$ on ∂M, $k_0 > 0$. Then

$$\nu_1 \ge k_0.$$

Equality holds only for Euclidean ball of radius k_0^{-1}.

We propose to prove the conjecture for rotationally invariant metrics on the ball.

2. Preliminaries

Throughout this paper B_r will be the n–dimensional ball of radius $r > 0$ parametrized by

$$X(s,w) = sY(w), \qquad 0 \le s \le r, \tag{3}$$

where $Y(w)$ is a standard parametrization of the unit sphere $(n-1)$–dimensional, S^{n-1}, given by

$$\begin{aligned} Y(w) &= Y(w_1,\dots,w_{n-1}) \\ &= (\sin w_{n-1}\sin w_{n-2}\cdots\sin w_1,\dots,\sin w_{n-1}\cos w_{n-2},\cos w_{n-1}). \end{aligned}$$

(B_r, g) will be the ball endowed with a rotationally invariant metric, i.e., such that in the parametrization (3) has the form

$$ds^2 + f^2(s)dw^2,$$

where dw^2 represents the standard metric on S^{n-1}, with $f(0) = 0$, $f'(0) = 1$ and $f(s) > 0$ for $0 < s \le r$.

If D is the Levi-Civita connection associated to the metric g, and $X_s(s,w) = Y(w)$, $X_i(s,w) = \frac{\partial}{\partial w_i}X(s,w) = sY_i(w)$, $i = 1,\dots,n-1$, are the coordinate fields corresponding to the parametrization (3), it is easy to verify the following identities:

$$D_{X_s}X_s = \overline{0}. \tag{4}$$

$$D_{X_i}X_s = \frac{f'}{f}X_i. \tag{5}$$

$$D_{X_{n-1}}X_{n-1} = -ff'X_s. \tag{6}$$

$$D_{X_{n-2}}X_{n-2} = -ff'\sin^2 w_{n-1}X_s - \sin w_{n-1}\cos w_{n-1}X_{n-1}. \tag{7}$$

$$D_{X_{n-1}}X_{n-2} = \frac{\cos w_{n-1}}{\sin w_{n-1}}X_{n-2}. \tag{8}$$

All calculations depend on the definition of the metric and the relation between the metric and the connection. To coordinate fields, this relation is given by

$$g\big(D_{X_i}X_j, X_k\big) = \frac{1}{2}X_j g\big(X_i, X_k\big) + \frac{1}{2}X_i g\big(X_j, X_k\big) - \frac{1}{2}X_k g\big(X_i, X_j\big). \tag{9}$$

These identities are necessary to calculate the mean curvature and the sectional curvatures. As an example we show the identity (8).

On the one hand

$$g\big(D_{X_{n-1}}X_{n-2}, X_s\big) = 0,$$

and

$$g\big(D_{X_{n-1}}X_{n-2}, X_i\big) = 0, \qquad i = 1,\dots,n-3.$$

On the other hand

$$\begin{aligned} g\big(D_{X_{n-1}}X_{n-2}, X_{n-1}\big) &= \frac{1}{2}X_{n-2}g\big(X_{n-1}, X_{n-1}\big) \\ &= \frac{1}{2}X_{n-2}f^2(s) = 0, \end{aligned}$$

and

$$\begin{aligned} g\big(D_{X_{n-1}}X_{n-2}, X_{n-2}\big) &= \frac{1}{2}X_{n-1}g\big(X_{n-2}, X_{n-2}\big) \\ &= \frac{1}{2}X_{n-1}f^2\sin^2 w_{n-1} \\ &= f^2 \sin w_{n-1}\cos w_{n-1}. \end{aligned}$$

From the above we conclude that $D_{X_{n-1}}X_{n-2} = \frac{\cos w_{n-1}}{\sin w_{n-1}}X_{n-2}$.

3. Curvatures

Let $X_s(r, w) = Y(w)$ be the outward normal vector field to the boundary of B_r, ∂B_r. The identity (5) implies that $g\big(D_{X_i}X_s, X_i\big) = g\Big(\frac{f'}{f}X_i, X_i\Big) = \Big(\frac{f'}{f}\Big)g\big(X_i, X_i\big)$. Hence, the mean curvature $h(r)$ is given by

$$h(r) = \frac{1}{n-1}\sum_{i=1}^{n-1} g\big(X_i, X_i\big)^{-1} g\big(D_{X_i}X_s, X_i\big) = \frac{f'(r)}{f(r)}. \tag{10}$$

Proposition 3.1. *The sectional curvature* $K\big(X_i, X_s\big)$, $i = 1, \dots, n-1$ *is given by*

$$K\big(X_i, X_s\big) = \frac{-f''(s)}{f(s)}. \tag{11}$$

Proof. From (5), $g\big(D_{X_i}X_s, X_i\big) = ff'dw^2\big(Y_i, Y_i\big)$. Differentiating with respect to X_s we get

$$\begin{aligned} g\big(D_{X_s}D_{X_i}X_s, X_i\big) &= -g\big(D_{X_i}X_s, D_{X_i}X_s\big) + \big(f'\big)^2 dw^2\big(Y_i, Y_i\big) + ff''dw^2\big(Y_i, Y_i\big) \\ &= -g\bigg(\frac{f'}{f}X_i, \frac{f'}{f}X_i\bigg) + \big(f'\big)^2 dw^2\big(Y_i, Y_i\big) + ff''dw^2\big(Y_i, Y_i\big) \\ &= ff''dw^2\big(Y_i, Y_i\big). \end{aligned}$$

From (4),

$$g\big(D_{X_i}D_{X_s}X_s, X_i\big) = g\big(D_{X_i}\overline{0}, X_i\big) = 0,$$

then

$$\begin{aligned} K\big(X_i, X_s\big) &= \frac{1}{g(X_i, X_i)}\big\{g\big(D_{X_i} D_{X_s} X_s, X_i\big) - g\big(D_{X_s} D_{X_i} X_s, X_i\big)\big\} \\ &= \frac{-f''}{f}. \end{aligned}$$ ☑

Proposition 3.2. *The sectional curvature* $K\big(X_{n-1}, X_{n-2}\big)$ *is given by*

$$K\big(X_{n-1}, X_{n-2}\big) = \frac{1 - \big(f'(s)\big)^2}{f^2(s)}. \tag{12}$$

***Proof*.** From identities (6),(7) and (8), we get

$$g\big(D_{X_{n-1}} X_{n-1}, D_{X_{n-2}} X_{n-2}\big) = \big(ff'\big)^2 \sin^2 w_{n-1},$$

and

$$g\big(D_{X_{n-1}} X_{n-2}, D_{X_{n-1}} X_{n-2}\big) = f^2 \cos^2 w_{n-1}.$$

From equation (9) it follows that

$$g\big(D_{X_{n-2}} X_{n-2}, X_{n-1}\big) = -\frac{1}{2} X_{n-1} g\big(X_{n-2}, X_{n-2}\big) = -f^2 \sin w_{n-1} \cos w_{n-1}.$$

Differentiating with respect to X_{n-1} we get

$$\begin{aligned} &g\big(D_{X_{n-1}} D_{X_{n-2}} X_{n-2}, X_{n-1}\big) = \\ &\qquad - g\big(D_{X_{n-2}} X_{n-2}, D_{X_{n-1}} X_{n-1}\big) + f^2\big(\sin^2 w_{n-1} - \cos^2 w_{n-1}\big). \end{aligned}$$

Therefore

$$\begin{aligned} &g(D_{X_{n-1}} D_{X_{n-2}} X_{n-2}, X_{n-1}) = \\ &\qquad - (ff')^2 \sin^2 w_{n-1} + f^2(\sin^2 w_{n-1} - \cos^2 w_{n-1}). \end{aligned} \tag{13}$$

On the other hand,

$$g\big(D_{X_{n-1}} X_{n-2}, X_{n-1}\big) = \frac{1}{2} X_{n-2} g\big(X_{n-1}, X_{n-1}\big) = 0.$$

Differentiating this equation with respect to X_{n-2} we get

$$\begin{aligned} &g(D_{X_{n-2}} D_{X_{n-1}} X_{n-2}, X_{n-1}) = \\ &\qquad - g(D_{X_{n-1}} X_{n-2}, D_{X_{n-1}} X_{n-2}) = -f^2 \cos^2 w_{n-1}. \end{aligned} \tag{14}$$

From equations (13) and (14) the proposition follows. ☑

4. The First Nonconstant Eigenfunction for the Stekloff Problem on B_r

In the following theorem Escobar characterized the first eigenfunction of a geodesic ball which has a rotationally invariant metric (see [4]).

Theorem 4.1. *Let B_r be a ball in $\mathbb{R}^n$ endowed with a rotationally invariant metric $ds^2 + f^2(s)dw^2$, where dw^2 represents the standard metric on S^{n-1}, with $f(0) = 0$, $f'(0) = 1$ and $f(s) > 0$ for $0 < s \leq r$. The first non–constant eigenfunction for the Stekloff problem on B_r has the form*

$$\varphi(s, w) = \psi(s)e(w), \tag{15}$$

where $e(w)$ satisfies the equation $\Delta e + (n-1)e = 0$ on S^{n-1} and the function ψ satisfies the differential equation

$$\frac{1}{f^{n-1}(s)}\frac{d}{ds}\left(f^{n-1}(s)\frac{d}{ds}\psi(s)\right) - \frac{(n-1)\psi(s)}{f^2(s)} = 0 \quad \text{in} \quad (0, r), \tag{16}$$

with the conditions

$$\begin{aligned} \psi'(r) &= \nu_1\psi(r), \\ \psi(0) &= 0. \end{aligned} \tag{17}$$

Proof. We use separation of variables and observe that the space $L^2(B_r)$ is equal to the space $L^2(0, r) \otimes L^2(S^{n-1})$. Let $\{e_i\}$, $i = 0, 1, 2, \ldots$, be a complete orthogonal set of eigenfunctions for the Laplacian on S^{n-1} with associated eigenvalues λ_i such that $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$. For $i \geq 1$, let ψ_i be the function satisfying

$$\frac{1}{f^{n-1}(s)}\frac{d}{ds}\left(f^{n-1}(s)\frac{d}{ds}\psi_i(s)\right) - \frac{(n-1)\psi_i(s)}{f^2(s)} = 0 \quad \text{in} \quad (0, r),$$

$\psi_i'(r) = \beta_i\psi_i(r)$, $\psi_i(0) = 0$.

Let $u_0 = 1$ and $u_i = \psi_i(s)e_i(w)$ for $i = 1, 2, \ldots$. The set $\{u_i\}$ for $i = 0, 1, 2, \ldots$ forms an orthogonal basis for $L^2(B_r)$.

Recall that the first non-zero Stekloff eigenvalue has the variational characterization

$$\nu_1 = \min_{\int_{\partial B_r} \varphi d\sigma = 0} \frac{\int_{B_r} \left|\nabla\varphi\right|^2 dv}{\int_{\partial B_r} \varphi^2 \, d\sigma}.$$

Since for $i \geq 1$

$$\begin{aligned} \beta_i &= \frac{\int_{B_r} \left|\nabla u_i\right|^2 f^{n-1} \, ds \, dw}{\int_{\partial B_r} u_i^2 f^{n-1} \, dw} \\ &= \frac{\int_0^r \left(\frac{d}{ds}\psi_i\right)^2 f^{n-1} \, ds + \lambda_i \int_0^r (\psi_i)^2 f^{n-3} \, ds}{\psi_i^2(r) f^{n-1}(r)} \end{aligned}$$

and $\lambda_i \geq \lambda_1 = n-1$, we get that $\beta_i \geq \beta_1$. Because the competing functions in the variational characterization of ν_1 are orthogonal to the constant functions on ∂B_r, we easily find that $\nu_1 = \beta_1$.

Using the formula $\Delta_g \varphi = \frac{\partial^2 \varphi}{\partial s^2} + (n-1)\frac{f'}{f}\frac{\partial \varphi}{\partial s} + \frac{1}{f^2}\Delta\varphi$, where Δ is the standard Laplacian on S^{n-1}, the equation (16) follows. ☑

When $n = 2$, the function ψ has the form $\psi(s) = ce^{\int^s \frac{du}{f(u)}}$ for c constant. The first eigenvalue and the mean curvature are given by $\nu_1 = \frac{\psi'(r)}{\psi(r)} = \frac{1}{f(r)}$ and $h(r) = \frac{f'(r)}{f(r)}$. From this we observe:

***Remark* 4.2.** When $f(s) = s + s^3$ or $f(s) = \sinh(s)$ (the hyperbolic space with curvature -1) since $f'(r) > 1$ then $\nu_1 < h(r)$. Therefore for $n = 2$, the condition that B_r has non-negative sectional curvature is necessary.

***Remark* 4.3.** From Proposition 3.1, the condition of non–negative sectional curvature implies that $f''(s) \leq 0$, and therefore f' is decreasing. Since $f'(0) = 1$, then $f'(r) \leq 1$. Hence, for $n = 2$ the condition of non–negative sectional curvature implies $\nu_1 \geq h(r)$. As examples of these metrics we have $f(s) = s$ (standard metric), $f(s) = \sin(s)$ (constant sectional curvature equal to 1) and $f(s) = s - \frac{s^3}{6}$.

5. Main Theorem

Theorem 5.1. *Let (B_r, g) be a ball in $\mathbb{R}^n$ $(n \geq 3)$ endowed with a rotationally invariant metric. Assume that B_r has non-negative sectional curvature and mean curvature on ∂B_r, $h(r) > 0$. Then the first non-zero eigenvalue of the Stekloff problem ν_1 satisfies $\nu_1 \geq h(r)$. Equality holds only for the standard metric of $\mathbb{R}^n$.*

***Proof*.** The coordinate functions are eigenfunctions of the Laplacian on S^{n-1}. From the equation (15) it follows that $\varphi(s, w) = \psi(s)\cos w_{n-1}$ is an eigenfunction associated to the first eigenvalue ν_1. Consider the function $F = \frac{1}{2}\left|\nabla\varphi\right|^2$. Since φ is a harmonic function and $\mathrm{Ric}\left(\nabla\varphi, \nabla\varphi\right) \geq 0$, the Weizenböck formula (see [5])

$$\Delta F = \left|\,\mathrm{Hess}(\varphi)\right|^2 + g\big(\nabla\varphi, \nabla(\Delta\varphi)\big) + \mathrm{Ric}(\nabla\varphi, \nabla\varphi)$$

implies that $\Delta F \geq 0$, and hence F is a subharmonic function. Therefore, the maximum of F is achieved at some point $P(r,\theta) \in \partial B_r$. Hopf's Maximum Principle implies that $\frac{\partial F}{\partial s}(r,\theta) > 0$ or F is constant. Since

$$\begin{aligned} F(s,w) &= \frac{1}{2}\left\{\left(\frac{\partial\varphi}{\partial s}\right)^2 + f^{-2}\left(\frac{\partial\varphi}{\partial w_{n-1}}\right)^2\right\} \\ &= \frac{1}{2}\left\{\left(\psi'\right)^2\cos^2\, w_{n-1} + \left(\frac{\psi}{f}\right)^2\sin^2\, w_{n-1}\right\} \end{aligned}$$

and F is a non-constant function, then

$$\frac{\partial F}{\partial s}(r,\theta) = \psi'\psi'' \cos^2\theta_{n-1} + \frac{\psi}{f}\left(\frac{\psi}{f}\right)' \sin^2\theta_{n-1} > 0. \tag{18}$$

Evaluating $\frac{\partial F}{\partial w_{n-1}}(s,w)$ at the point P we find that

$$\frac{\partial F}{\partial w_{n-1}}(r,\theta) = \left(\left(\frac{\psi}{f}\right)^2 - \big(\psi'\big)^2\right)\sin\theta_{n-1}\cos\theta_{n-1} = 0. \tag{19}$$

The equation (19) implies that

$$\left(\frac{\psi(r)}{f(r)}\right)^2 - \big(\psi'(r)\big)^2 = 0,$$

or

$$\sin\theta_{n-1} = 0 \quad \text{and} \quad \cos^2\theta_{n-1} = 1,$$

or

$$\sin^2\theta_{n-1} = 1 \quad \text{and} \quad \cos\theta_{n-1} = 0.$$

If

$$\left(\frac{\psi(r)}{f(r)}\right)^2 - \big(\psi'(r)\big)^2 = 0,$$

given that $\psi(r) \neq 0$ ($\psi(r) = 0$ implies $\varphi = 0$ on ∂B_r and thus, φ is a constant function on B_r which is a contradiction), it follow from (17) that

$$(\nu_1)^2 = \left(\frac{\psi'(r)}{\psi(r)}\right)^2 = \left(\frac{1}{f(r)}\right)^2. \tag{20}$$

The condition $h(r) > 0$ and (10) implies that $f'(r) > 0$. Since B_r has non–negative sectional curvature then (12) implies that $1 \geq (f')^2$. Then

$$\left(\frac{1}{f(r)}\right)^2 \geq \left(\frac{f'(r)}{f(r)}\right)^2 = \big(h(r)\big)^2. \tag{21}$$

From (20) and (21) it follows that $\nu_1 \geq h(r)$.

Equality holds only for $f'(r) = 1$. If

$$\sin\theta_{n-1} = 0 \quad \text{and} \quad \cos^2\theta_{n-1} = 1,$$

then

$$F\big(r,\theta_1,\dots,\theta_{n-2},\theta_{n-1}\big) - F\left(r,\theta_1,\dots,\theta_{n-2},\frac{\pi}{2}\right) = \frac{1}{2}\left\{(\psi')^2 - \left(\frac{\psi}{f}\right)^2\right\} \geq 0$$

thus

$$(\nu_1)^2 = \left(\frac{\psi'(r)}{\psi(r)}\right)^2 \geq \left(\frac{1}{f(r)}\right)^2 \geq \left(\frac{f'(r)}{f(r)}\right)^2 = \big(h(r)\big)^2.$$

Equality holds only for $f'(r) = 1$.

If

$$\sin^2\theta_{n-1} = 1 \quad \text{and} \quad \cos\theta_{n-1} = 0,$$

from (18) we have

$$\frac{\partial F}{\partial s}(P) > 0$$

since

$$\frac{\psi}{f}\left(\frac{\psi}{f}\right)' > 0.$$

Thus

$$\left(\frac{\psi}{f}\right)\left(\frac{f\nu_1\psi - f'\psi}{f^2}\right) = \left(\frac{\psi}{f}\right)^2\big(\nu_1 - h(r)\big) > 0.$$

The inequality is strict.

In any case we conclude that $\nu_1 \geq h(r)$. If equality is attained then $f'(r) = 1$. Since the sectional curvature is non-negative, then (11) implies that $f''(s) \leq 0$. $f'(0) = 1 = f'(r)$ and $f''(s) \leq 0$ implies $f' \equiv 1$. Since $f(0) = 0$, then $f(s) = s$. Consequently g is the standard metric on $\mathbb{R}^n$. ☑

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