Revista Colombiana de Matemáticas
Volumen 47(2013)1, páginas 39-66

The Diagonal General Case of the Laguerre-Sobolev Type Orthogonal Polynomials

El caso general diagonal de los polinomios ortogonales de tipo Laguerre-Sobolev

Herbert Dueñas[1,✉], Juan C. García[1], Luis E. Garza[2], Alejandro Ramírez[2]

[1]Universidad Nacional de Colombia, Bogotá, Colombia

[2]Universidad de Colima, Colima, México

Abstract. We consider the family of polynomials orthogonal with respect to the Sobolev type inner product corresponding to the diagonal general case of the Laguerre-Sobolev type orthogonal polynomials. We analyze some properties of these polynomials, such as the holonomic equation that they satisfy and, as an application, an electrostatic interpretation of their zeros. We also obtain a representation of such polynomials as a hypergeometric function, and study the behavior of their zeros.

Key words and phrases. Orthogonal polynomials, Laguerre-Sobolev type polynomials, Laguerre Polynomials, Derivative of a Dirac Delta.

2010 Mathematics Subject Classification. 33C47, 42C05.

Resumen. Se considera la familia de los polinomios ortogonales con respecto a un producto interno de tipo Sobolev correspondiente al caso general diagonal de los polinomios ortogonales de tipo Laguerre-Sobolev. Se analizan algunas propiedades de estos polinomios tales como la ecuación holonómica que satisfacen y, como una aplicación de dicha ecuación, una interpretación electrostática de sus ceros. También se obtiene una representación de tales polinomios en términos de una función hipergeométrica, y se estudia el comportamiento de sus ceros.

Palabras y frases clave. Polinomios ortogonales, polinomios de tipo Laguerre-Sobolev, polinomios de Laguerre, derivada de una Delta de Dirac.

1. Introduction

The Laguerre polynomials are defined as the orthogonal polynomials with respect to the inner product

$$\langle p, q\rangle_\alpha = \int_I p(x)q(x)\, d\sigma(x), \tag{1}$$

where $d\sigma(x) = x^\alpha e^{-x}dx$, I is the positive real line, $\alpha > -1$, and p, q are polynomials with real coefficients.

We denote by $\left\{L_n^\alpha(x)\right\}_{n=0}^\infty$ the sequence of monic Laguerre polynomials, and by $\left\{\widetilde{L}_n^\alpha(x)\right\}_{n=0}^\infty$ the sequence of orthogonal polynomials with respect to the inner product

$$\langle p, q\rangle_{S,\alpha} = \int_I p(x)q(x)\, d\sigma + \sum_{i=0}^{j} M_i p^{(i)}(0)q^{(i)}(0), \tag{2}$$

where $M_i \in \mathbb{R}_+$ and $j \in \mathbb{N}$. The goal of this paper is to generalize some results obtained in [4], [5] and [10], extending them for an arbitrary finite number of masses.

The structure of the manuscript is as follows. In Section 2, we present some preliminary results about the Laguerre polynomials. In Section 3, we give a connection formula which expresses the Laguerre-Sobolev type polynomials in terms of some Laguerre polynomials. This formula will be our principal tool.

A $2j + 2$ terms recurrence relation for the Laguerre-Sobolev type polynomials is deduced in Section 4. In Section 5, we show a second order differential equation that the Laguerre-Sobolev type polynomials satisfy. This holonomic equation will be used in Section 6 in order to find an electrostatic interpretation of the zeros of $\widetilde{L}_n^\alpha(x)$.

In section 7, we use the hypergeometric representation of the Laguerre polynomials to construct a hypergeometric representation for the Laguerre-Sobolev type polynomials and finally, in Section 8, we analyze the location of the zeros of $\widetilde{L}_n^\alpha(x)$. Some numerical experiments using MatLab and Mathematica software are presented.

2. Preliminaries

Let μ be a linear moment functional defined in $\mathbb{P}$, the linear space of polynomials with real coefficients, and define their corresponding moments by $\mu_n = \langle \mu, x^n\rangle$, $n \geq 0$. If $\{P_n(x)\}_{n\geq 0}$ is a polynomials sequence such that the degree of $P_n(x)$ is n and it satisfies

$$\big\langle \mu, P_n(x)P_m(x)\big\rangle = K_n\delta_{m,n},$$

with $K_n \neq 0$, $n, m \in \mathbb{N}$, and $\delta_{m,n}$ is the Kronecker function defined by

$$\delta_{m,n} = \begin{cases} 0, & \text{if} \quad m \neq n; \\ 1, & \text{if} \quad n = m, \end{cases}$$

then we say that $\{P_n(x)\}_{n\geq 0}$ is the sequence of orthogonal polynomials associated with the linear functional μ. We will assume that μ is a positive definite linear functional, i.e. $\langle \mu, \pi(x) \rangle > 0$ for any polynomial $\pi(x) > 0$. In such a case, μ has an integral representation in terms of a positive measure, and the orthogonality relation can be expressed in terms of an inner product such as (1).

It is well known that $\{P_n(x)\}_{n\geq 0}$ satisfies the following three term recurrence relation:

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma P_{n-1}(x),$$

where $P_{-1}(x) = 0$, $P_0(x) = 1$ and, for $n \in \mathbb{N}$, the recurrence coefficients $\{\beta_n\}_{n\in\mathbb{N}}$, $\{\gamma_n\}_{n\in\mathbb{N}}$ are real with $\gamma_n \neq 0$. The n-th reproducing Kernel $K_n(x, y)$ associated with $\{P_n(x)\}_{n\geq 0}$ is defined by

$$K_n(x, y) = \sum_{k=0}^{n} \frac{P_k(x)P_k(y)}{\|P_k\|_\mu^2},$$

where $\|P_k\|_\mu^2 = \left\langle \mu, \big(P(x)\big)^2 \right\rangle$.

There is an explicit expression for $K_n(x, y)$, the so-called Christoffel-Darboux formula, given by

$$K_n(x, y) = \frac{P_{n+1}(x)P_n(y) - P_n(x)P_{n+1}(y)}{(x - y)\|P_n\|_\mu^2},$$

for $x \neq y$. We will use the following notation for the partial derivatives of $K_n(x, y)$

$$\frac{\partial^{i+t}\big(K_n(x, y)\big)}{\partial x^i \partial y^t} = K_n^{(i,t)}(x, y).$$

If $p \in \mathbb{P}$ and $\deg(p) \leq n$ then we have

$$\left\langle K_n^{(0,i)}(x, y), p(x) \right\rangle = p^{(i)}(y), \tag{3}$$

which is called the reproducing property.

In [4], the Leibniz formula for the j-th derivative of a product is used to deduce the following formula:

$$K_{n-1}^{(0,j)}(x,a) = \frac{j!}{\|P_{n-1}\|^2(x-a)^{j+1}} \times$$
$$\left(P_n(a) + P_n^{(1)}(a)(x-a) + \frac{P_n^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{P_n^{(n)}(a)}{n!}(x-a)^n\right) \times$$
$$\left(P_{n-1}(a) + P_{n-1}^{(1)}(a)(x-a) + \frac{P_{n-1}^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{P_{n-1}^{(j)}(a)}{j!}(x-a)^j\right) -$$
$$\frac{j!}{\|P_{n-1}\|^2(x-a)^{j+1}} \times$$
$$\left(P_{n-1}(a) + P_{n-1}^{(1)}(a)(x-a) + \frac{P_{n-1}^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{P_{n-1}^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1}\right)$$
$$\times \left(P_n(a) + P_n^{(1)}(a)(x-a) + \frac{P_n^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{P_n^{(j)}(a)}{j!}(x-a)^j\right). \quad (4)$$

In the following section, we will deduce a formula for $K_{n-1}^{(i,j)}(a,a)$, which will help us to deduce the derivative of the polynomial $\widetilde{L}_n^{\alpha}(x)$.

The families of orthogonal polynomials that have been most extensively studied in the literature are the Jacobi, Laguerre and Hermite polynomials. They are the so-called classical orthogonal polynomials, and have (among others) the following characterization:

$\{P_n(x)\}_{n\in\mathbb{N}}$ is classical if and only if P_n, $n \in \mathbb{N}$, is a solution of the second order differential equation

$$\phi(x)y''(x) + \tau(x)y'(x) + \lambda_n y(x) = 0,$$

where ϕ and τ are polynomials independent of n, with degree at most 2 and exactly 1, respectively, and λ_n is a constant.

In this paper, we will focus our attention on the *Laguerre* monic orthogonal polynomials. In the following proposition we list some of their properties, that will be useful in the upcoming sections (see [2], [3] and [12]).

Proposition 1. *Let $\{L_n^{\alpha}\}_{n\geq 0}$ be the sequence of Laguerre monic orthogonal polynomials. Then,*

1) For $n \in \mathbb{N}$,

$$xL_n^{\alpha}(x) = L_{n+1}^{\alpha}(x) + (2n+\alpha+1)L_n^{\alpha}(x) + n(n+\alpha)L_{n-1}^{\alpha}(x). \quad (5)$$

2) For $n \in \mathbb{N}$,

$$L_n^\alpha(x) = (-1)^n(\alpha+1)_n \sum_{k=0}^{\infty} \frac{(-n)_k}{(\alpha+1)_k} \frac{x^k}{k!} = \\ (-1)^n(\alpha+1)_n \times {}_1F_1(-n;\alpha+1), \quad (6)$$

where

$$(a)_m := \begin{cases} 1, & \text{if} \quad m = 0; \\ \prod_{k=0}^{|m-1|} (a + k\,\mathrm{sgn}(m)), & \text{if} \quad m \in \mathbb{Z} \setminus \{0\} \end{cases}$$

and

$$\mathrm{sgn}(m) := \begin{cases} 1, & \text{if} \quad m \in \mathbb{Z}_+; \\ -1, & \text{if} \quad m \in \mathbb{Z}_-. \end{cases}$$

In other words, the Laguerre polynomials can be expressed as a hypergeometric function of the form ${}_1F_1$.

3) *For* $n \in \mathbb{N}$,

$$(L_n^\alpha)^{(i)}(0) = (-1)^{n+i} \frac{n!\Gamma(\alpha+n+1)}{(n-i)!\Gamma(\alpha+i+1)}. \quad (7)$$

4) *For* $n \in \mathbb{N}$,

$$\|L_n^\alpha\|_\alpha^2 = n!\Gamma(n+\alpha+1). \quad (8)$$

5) *For* $n \in \mathbb{N}$, $L_n^\alpha(x)$ *satisfies the differential equation*

$$xy'' + (\alpha+1-x)y' = -ny. \quad (9)$$

6) *For* $n \in \mathbb{N}$,

$$x\big(L_n^\alpha(x)\big)' = nL_n^\alpha(x) + n(n+\alpha)L_{n-1}^\alpha(x). \quad (10)$$

7) *For* $n \in \mathbb{N}$,

$$L_n^\alpha(x) = \frac{n!}{(-1)^n} \sum_{k=0}^{n} \frac{\Gamma(n+\alpha+1)}{(n-k)!\Gamma(\alpha+k+1)} \frac{(-x)^k}{k!}. \quad (11)$$

8) *For* $n \in \mathbb{N}$,

$$L_n^\alpha(x) = L_n^{\alpha+1}(x) + nL_{n-1}^{\alpha+1}(x). \quad (12)$$

9) *For* $n \in \mathbb{N}$,

$$\big(L_n^\alpha\big)'(x) = nL_{n-1}^{\alpha+1}(x). \quad (13)$$

3. Connection Formula

Let $\{P_n\}_{n\geq 0}$ be the sequence of monic orthogonal polynomials with respect to a moment linear functional μ, and denote by $\left\{\widetilde{P}_n(x)\right\}_{n\geq 0}$ the sequence of polynomials orthogonal with respect to the inner product

$$\langle p,q\rangle_S = \langle p,q\rangle_\mu + \sum_{i=0}^{j} M_i\, p^{(i)}(0)\, q^{(i)}(0).$$

Since $\{P_n\}_{n\geq 0}$ is a basis in the linear space of polynomials, we can write

$$\widetilde{P}_n(x) = \sum_{k=0}^{n} a_{n,k} P_k(x) = P_n(x) + \sum_{k=0}^{n-1} a_{n,k} P_k(x), \tag{14}$$

where

$$a_{n,k} = \frac{\left\langle \widetilde{P}_n(x), P_k(x)\right\rangle_\mu}{\|P_k(x)\|_\mu^2}.$$

Taking into account that for $k < n$

$$\left\langle \widetilde{P}_n(x), P_k(x)\right\rangle_\mu = -\sum_{i=0}^{j} M_i \widetilde{P}_n^{(i)}(a) P_k^{(i)}(a),$$

$\widetilde{P}_n(x)$ can be written as

$$\widetilde{P}_n(x) = P_n(x) - \sum_{i=0}^{j} M_i \widetilde{P}_n^{(i)}(a) \sum_{k=0}^{n-1} \frac{P_k^{(i)}(a) P_k(x)}{\|P_k(x)\|_\mu^2} \tag{15}$$

$$= P_n(x) - \sum_{i=0}^{j} M_i \widetilde{P}_n^{(i)}(a) K_{n-1}^{(0,i)}(x,a). \tag{16}$$

In order to find $\widetilde{P}_n^{(i)}(a)$ for $i = 0,\dots,j$, we need to solve the linear system,

$$\begin{bmatrix} 1+M_0K^{(0,0)} & M_1K^{(0,1)} & \cdots & M_jK^{(0,j)} \\ M_0K^{(1,0)} & 1+M_1K^{(1,1)} & \cdots & M_jK^{(1,j)} \\ M_0K^{(2,0)} & M_1K^{(2,1)} & \cdots & M_jK^{(2,j)} \\ \vdots & \vdots & \ddots & \vdots \\ M_0K^{(j,0)} & M_1K^{(j,1)} & \cdots & 1+M_jK^{(j,j)} \end{bmatrix} \begin{bmatrix} \widetilde{P}_n^{(0)}(a) \\ \widetilde{P}_n^{(1)}(a) \\ \widetilde{P}_n^{(2)}(a) \\ \vdots \\ \widetilde{P}_n^{(j)}(a) \end{bmatrix} = \begin{bmatrix} P_n^{(0)}(a) \\ P_n^{(1)}(a) \\ P_n^{(2)}(a) \\ \vdots \\ P_n^{(j)}(a) \end{bmatrix} \tag{17}$$

which is obtained by taking derivatives in (16), where $K^{(p,q)} := K_{n-1}^{(p,q)}(a,a)$. According to (4) we have

$$K_{n-1}^{(0,j)}(x,a) = \frac{j!}{\|P_{n-1}\|^2} \times \Bigg[\sum_{r=j+1}^{n} \frac{P_n^{(r)}(a)}{r!} \Bigg(P_{n-1}(a)(x-a)^{r-j-1} + \cdots + \frac{P_{n-1}^{(k)}(a)}{k!}(x-a)^{k+r-j-1} + \cdots + \frac{P_{n-1}^{(j)}(a)}{j!}(x-a)^{r-1} \Bigg) - \sum_{k=j+1}^{n-1} \frac{P_{n-1}^{(k)}(a)}{k!} \Bigg(P_n(a)(x-a)^{k-j-1} + \cdots + \frac{P_n^{(r)}(a)}{r!}(x-a)^{k+r-j-1} + \cdots + \frac{P_n^{(j)}(a)}{j!}(x-a)^{k-1} \Bigg) \Bigg]. \quad (18)$$

Having in mind that we are interested in finding $K_{n-1}^{(i,j)}(a,a)$, we calculate the i-th derivative in (18) with respect to x and evaluate it at $x = a$ to obtain

$$K_{n-1}^{(i;j)}(a,a) = \frac{j!}{\|P_{n-1}\|^2} \Bigg[\sum_{r=j+1}^{n} i! \frac{P_n^{(r)}(a) P_{n-1}^{(i+j+1-r)}(a)}{r!(i+j+1-r)!} - \sum_{k=j+1}^{n-1} i! \frac{P_n^{(i+j+1-k)}(a) P_{n-1}^{(k)}(a)}{k!(i+j+1-k)!} \Bigg]. \quad (19)$$

Our goal is to write the polynomials $\widetilde{L}_n^{\alpha}(x)$, orthogonal with respect to the inner product (2), in terms of some Laguerre orthogonal polynomials. First, we need the following theorem.

Theorem 2. *For $t, n \in \mathbb{N}$ and $\alpha > -1$, the Laguerre polynomials satisfy the following property ([9])*

$$L_n^{\alpha}(x) = \sum_{k=0}^{t} \binom{t}{k} (n)_{-k} L_{n-k}^{\alpha+t}(x). \quad (20)$$

Let $\{L_n^{\alpha+j+1}(x)\}$ be the sequence of Laguerre polynomials orthogonal with respect to $d\sigma(x) = e^{-x} x^{\alpha+j+1}\, dx$. We will express $K_{n-1}^{(0,i)}(x,0)$ as a linear combination of $L_n^{\alpha+j+1}(x)$. Then,

$$\frac{\|L_{n-1}^{\alpha}\|_{\alpha}^2}{(L_{n-1}^{\alpha})^{(i)}(0)} K_{n-1}^{(0,i)}(x,0) = L_{n-1}^{\alpha+j+1}(x) + \sum_{k=0}^{n-2} b_{n-1,k} L_k^{\alpha+j+1}(x), \quad (21)$$

where

$$
\begin{aligned}
b_{n-1,k} &= \frac{\left\langle \frac{\|L_{n-1}^{\alpha}\|_{\alpha}^{2}}{(L_{n-1}^{\alpha})^{(i)}(0)} K_{n-1}^{(0,i)}(x,0), L_k^{\alpha+j+1}(x) \right\rangle_{\alpha+j+1}}{\|L_k^{\alpha+j+1}(x)\|_{\alpha+j+1}^{2}} \\
&= \frac{\|L_{n-1}^{\alpha}\|_{\alpha}^{2}}{(L_{n-1}^{\alpha})^{(i)}(0)\|L_k^{\alpha+j+1}(x)\|_{\alpha+j+1}^{2}} \left\langle K_{n-1}^{(0,i)}(x,0), L_k^{\alpha+j+1}(x) \right\rangle_{\alpha+j+1}.
\end{aligned}
$$

From (1) and (3), we see that for $0 \leq k \leq n-2-j$,

$$
\begin{aligned}
\left\langle K_{n-1}^{(0,i)}(x,0), L_k^{\alpha+j+1}(x) \right\rangle_{\alpha+j+1} &= \int_0^{\infty} K_{n-1}^{(0,i)}(x,0) L_k^{\alpha+j+1}(x) x^{\alpha+j+1} e^{-x}\, dx \\
&= \int_0^{\infty} K_{n-1}^{(0,i)}(x,0)(x^{j+1} L_k^{\alpha+j+1}(x)) x^{\alpha} e^{-x}\, dx \\
&= \left\langle K_{n-1}^{(0,i)}(x,0), x^{j+1} L_k^{\alpha+j+1}(x) \right\rangle_{\alpha} \\
&= \left(x^{j+1} L_k^{\alpha+j+1}(x) \right)^{(i)}(0) = 0.
\end{aligned}
$$

In other words,

$$
\begin{aligned}
K_{n-1}^{(0,i)}(x,0) = \frac{(L_{n-1}^{\alpha})^{(i)}(0)}{\|L_{n-1}^{\alpha}(x)\|_{\alpha}^{2}} L_{n-1}^{\alpha+j+1}(x) + \\
\sum_{k=n-j-1}^{n-2} \frac{\left\langle K_{n-1}^{(0,i)}(x,0) L_k^{\alpha+j+1}(x) \right\rangle_{\alpha+j+1}}{\|L_k^{\alpha+j+1}(x)\|_{\alpha+j+1}^{2}} L_k^{\alpha+j+1}(x). \quad (22)
\end{aligned}
$$

Taking into account that

$$
K_{n-1}^{(0,i)}(x,0) = \sum_{m=0}^{n-1} \frac{L_m^{\alpha}(x) \left(L_m^{\alpha}\right)^{(i)}(0)}{\|L_m^{\alpha}(x)\|_{\alpha}^{2}}, \qquad (23)
$$

and using (7), (8) and (20), we have

$$\begin{aligned}
&\left\langle K_{n-1}^{(0,i)}(x,0), L_k^{\alpha+j+1}(x)\right\rangle_{\alpha+j+1} \\
&= \sum_{m=0}^{n-1} \frac{(L_m^{\alpha})^{(i)}(0)}{\|L_m^{\alpha}(x)\|_{\alpha}^2} \left\langle L_m^{\alpha}(x), L_k^{\alpha+j+1}(x)\right\rangle_{\alpha+j+1} \\
&= \sum_{m=0}^{n-1} \frac{(L_m^{\alpha})^{(i)}(0)}{\|L_m^{\alpha}(x)\|_{\alpha}^2} \sum_{t=0}^{j+1} \binom{j+1}{t} (m)_{(-t)} \left\langle L_{m-t}^{\alpha+j+1}(x), L_k^{\alpha+j+1}(x)\right\rangle_{\alpha+j+1} \\
&= \sum_{m=0}^{n-1} \sum_{t=0}^{j+1} \binom{j+1}{t} \frac{(m)_{(-t)} (L_m^{\alpha})^{(i)}(0)}{\|L_m^{\alpha}(x)\|_{\alpha}^2} \left\langle L_{m-t}^{\alpha+j+1}(x), L_k^{\alpha+j+1}(x)\right\rangle_{\alpha+j+1} \\
&= \sum_{m=i}^{n-1} \sum_{t=0}^{j+1} \binom{j+1}{t} \frac{(-1)^{m+i} m!}{(m-t)!\Gamma(\alpha+i+1)(m-i)!} \|L_k^{\alpha+j+1}(x)\|_{\alpha+j+1}^2 \, \delta_{k,m-t}.
\end{aligned}$$

For $n-j-1 \leq k \leq n-2$, let

$$c_{(k;i)} = \frac{\left\langle K_{n-1}^{(0,i)}(x,0), L_k^{\alpha+j+1}(x)\right\rangle_{\alpha+j+1}}{\|L_k^{\alpha+j+1}(x)\|_{\alpha+j+1}^2} \qquad \text{and} \qquad c_{(n-1;i)} = \frac{(L_{n-1},^{\alpha})^{(i)}(0)}{\|L_{n-1}^{\alpha}(x)\|_{\alpha}^2},$$

or, equivalently,

$$c_{(k;i)} = \sum_{m=i}^{n-1} \sum_{t=0}^{j+1} \binom{j+1}{t} \frac{(-1)^{m+i} m!}{(m-t)!\Gamma(\alpha+i+1)(m-i)!} \delta_{k,m-t}$$

and

$$c_{(n-1;i)} = \frac{(-1)^{n+i-1}}{(n-i-1)!\Gamma(\alpha+i+1)}. \tag{24}$$

Then,

$$\widetilde{P}_n(x) = L_n^{\alpha}(x) - \sum_{i=0}^{j} M_i \widetilde{P}_n^{(i)}(0) \sum_{k=n-j-1}^{n-1} c_{(k,i)} L_k^{\alpha+j+1}(x), \tag{25}$$

and using (20), we have

$$\begin{aligned}
\widetilde{L}_n^{\alpha}(x) = \sum_{k=0}^{j+1} \binom{j+1}{k} (n)_{-k} L_{n-k}^{\alpha+j+1}(x) - \\
\sum_{k=n-j-1}^{n-1} \left[\sum_{i=0}^{j} c_{(k,i)} M_i \left(\widetilde{L}_n^{\alpha}\right)^{(i)}(0) \right] L_k^{\alpha+j+1}(x).
\end{aligned}$$

As a consequence, we have the following theorem.

Theorem 3. *Let $\left\{L_t^{\alpha+j+1}(x)\right\}_{t=0}^{\infty}$ be the sequence Laguerre polynomials with parameter $\alpha+j+1$. Then, the Laguerre-Sobolev type polynomials can be written as*

$$\widetilde{L}_n^{\alpha}(x) = \sum_{k=0}^{j+1} A_n^{[k]} L_{n-k}^{\alpha+j+1}(x), \tag{26}$$

where, for $k = 1, 2, \ldots, j+1$, $A_n^{[k]}$ are constants of the form

$$A_n^{[k]} = -\sum_{i=0}^{j} c_{(n-k;i)} M_i \left(\widetilde{L}_n^{\alpha}\right)^{(i)}(0) + \binom{j+1}{k}(n)_{-k},$$

and $A_n^{[0]} = 1$.

4. The Recurrence Formula

The Fourier expansion of the polynomial $x^{j+1}\widetilde{L}_n^{\alpha}(x)$ is given by

$$x^{j+1}\widetilde{L}_n^{\alpha}(x) = \widetilde{L}_{n+j+1}^{\alpha}(x) + \sum_{k=0}^{n+j} \lambda_{n,k} \widetilde{L}_k^{\alpha}(x), \tag{27}$$

where

$$\lambda_{n,k} = \frac{\left\langle x^{j+1}\widetilde{L}_n^{\alpha}(x), \widetilde{L}_k^{\alpha}(x)\right\rangle_{S,\alpha}}{\|\widetilde{L}_k^{\alpha}\|_{S,\alpha}^2}.$$

Now, for each $p, q \in \mathbb{P}$ and each $i \in \{0, \ldots, j\}$, we have

$$\left(x^{j+1}p(x)\right)^{(i)}(0) = \sum_{s=0}^{i} \frac{(j+1)!}{(j+1-s)!}\binom{i}{s}\left[x^{j+1-s}p^{(i-s)}(x)\right]_{x=0} = 0$$

and thus

$$\left\langle x^{j+1}p(x), q(x)\right\rangle_{S,\alpha} = \left\langle p(x), q(x)\right\rangle_{\alpha+j+1}.$$

Furthermore,

$$\left\langle p(x), x^{j+1}q(x)\right\rangle_{S,\alpha} = \left\langle x^{j+1}p(x), q(x)\right\rangle_{S,\alpha}.$$

It follows from the previous observations that

$$\lambda_{n,k} = \frac{\left\langle \widetilde{L}_n^{\alpha}(x), x^{j+1}\widetilde{L}_k^{\alpha}(x)\right\rangle_{S,\alpha}}{\|\widetilde{L}_k^{\alpha}\|_{S,\alpha}^2} = \frac{\left\langle \widetilde{L}_n^{\alpha}(x), \widetilde{L}_k^{\alpha}(x)\right\rangle_{\alpha+j+1}}{\|\widetilde{L}_k^{\alpha}\|_{S,\alpha}^2}.$$

Therefore, for $k \in \{0, \ldots, n-(j+2)\}$, we have $\lambda_{n,k} = 0$. Using the previous equation and (26), we get

$$\begin{aligned}\left\langle \widetilde{L}_n^\alpha(x), \widetilde{L}_k^\alpha(x) \right\rangle_{\alpha+j+1} &= \sum_{s=0}^{j+1} A_n^{[s]} \left\langle L_{n-s}^{\alpha+j+1}(x), \widetilde{L}_k^\alpha(x) \right\rangle_{\alpha+j+1} \\ &= \sum_{s=0}^{j+1}\sum_{t=0}^{j+1} A_n^{[s]} A_k^{[t]} \left\langle L_{n-s}^{\alpha+j+1}(x), L_{k-t}^{\alpha+j+1}(x) \right\rangle_{\alpha+j+1} \\ &= \sum_{s=n-k}^{j+1} A_n^{[s]} A_k^{[k+s-n]} \left\| L_{n-s}^{\alpha+j+1} \right\|_{\alpha+j+1}^2\end{aligned}$$

for $k \in \{n-(j+1), n-j, \dots, n, \dots, n+j-1, n+j\}$. As a consequence, we have the following result.

Theorem 4. *For $n \in \mathbb{N}$, we have*

$$x^{j+1}\widetilde{L}_n^\alpha(x) = \widetilde{L}_{n+j+1}^\alpha(x) + \sum_{k=\max\{0,n-j-1\}}^{n+j} \lambda_{n,k}\widetilde{L}_k^\alpha(x),$$

where

$$\lambda_{n,k} = \sum_{s=n-k}^{j+1} A_n^{[s]} A_k^{[k+s-n]} \frac{\left\| L_{n-s}^{\alpha+j+1} \right\|_{\alpha+j+1}^2}{\left\| \widetilde{L}_k^\alpha \right\|_{S,\alpha}^2}.$$

5. Holonomic Equation

In order to find the second order differential equation satisfied by the Laguerre-Sobolev type orthogonal polynomials, we obtain from (9), for $0 \le k \le j+1$,

$$x\left(L_{n-k}^{\alpha+j+1}(x)\right)'' + (\alpha+j+2-x)\left(L_{n-k}^{\alpha+j+1}(x)\right)' = -(n-k)L_{n-k}^{\alpha+j+1}(x).$$

Thus,

$$xA_n^{[k]}\left(L_{n-k}^{\alpha+j+1}(x)\right)'' + (\alpha+j+2-x)A_n^{[k]}\left(L_{n-k}^{\alpha+j+1}(x)\right)' = -(n-k)A_n^{[k]}L_{n-k}^{\alpha+j+1}(x)$$

and, as a consequence, for $0 \le k \le j+1$,

$$\begin{aligned}\sum_{k=0}^{j+1}\left[xA_n^{[k]}\left(L_{n-k}^{\alpha+j+1}(x)\right)'' + (\alpha+j+2-x)A_n^{[k]}\left(L_{n-k}^{\alpha+j+1}(x)\right)'\right] = x\left(\widetilde{L}_n^\alpha(x)\right)'' \\ + (\alpha+j+2-x)\left(\widetilde{L}_n^\alpha(x)\right)' = -\sum_{k=0}^{j+1} A_n^{[k]}(n-k)L_{n-k}^{\alpha+j+1}(x). \quad (28)\end{aligned}$$

Our goal now is to express $L_{n-k}^{\alpha+j+1}(x)$ as a combination of $\left(\widetilde{L}_n^\alpha(x)\right)'$ and $\widetilde{L}_n^\alpha(x)$, with rational functions as coefficients. First, we will show that $L_{n-k}^{\alpha+j+1}(x)$ can be written as a combination of the Laguerre polynomials $L_n^{\alpha+j+1}(x)$ and $L_{n-1}^{\alpha+j+1}(x)$.

Theorem 5. *For each $k, n \in \mathbb{N}$ and $k \leq n$, we have*

$$L_{n-k}^{\alpha}(x) = D_{n-k}(x)L_{n-1}^{\alpha}(x) + S_{n-k}(x)L_{n}^{\alpha}(x), \tag{29}$$

where

$$D_{n-k}(x) = \frac{(x - M_{n-k+1})D_{n-k+1}(x)}{N_{n-k+1}} - \frac{D_{n-k+2}(x)}{N_{n-k+1}}$$

and

$$S_{n-k}(x) = \frac{(x - M_{n-k+1})S_{n-k+1}(x)}{N_{n-k+1}} - \frac{S_{n-k+2}(x)}{N_{n-k+1}},$$

with $D_{n-1}(x) = 1$, $D_n(x) = 0$, $S_{n-1}(x) = 0$, $S_n(x) = 1$, $M_{n-t} = \big(2(n-t) + 1 + \alpha\big)$ and $N_{n-t} = (n-t)(n-t+\alpha)$.

Proof. We will use induction on k. It is clear that the statement holds for $k = 1$. Assume that the property is valid for all $t \leq k$. Using (5), we get

$$xL_{n-k}^{\alpha}(x) = L_{n-k+1}^{\alpha}(x) + M_{n-k}L_{n-k}^{\alpha}(x) + N_{n-k}L_{n-k-1}^{\alpha}(x).$$

Thus,

$$\begin{aligned}
L_{n-k-1}^{\alpha}(x) = \frac{(x - M_{n-k})}{N_{n-k}}L_{n-k}^{\alpha}(x) - \frac{1}{N_{n-k}}L_{n-k+1}^{\alpha}(x) = \\
\frac{(x - M_{n-k})}{N_{n-k}}\Big[D_{n-k}(x)L_{n-1}^{\alpha}(x) + S_{n-k}(x)L_{n}^{\alpha}(x)\Big] - \\
\frac{1}{N_{n-k}}\Big[D_{n-k+1}(x)L_{n-1}^{\alpha}(x) + S_{n-k+1}(x)L_{n}^{\alpha}(x)\Big] = \\
\left[\frac{\big(x - M_{n-k}(x)\big)}{N_{n-k}}D_{n-k}(x) - \frac{D_{n-k+1}(x)}{N_{n-k}}\right]L_{n-1}^{\alpha}(x) + \\
\left[\frac{(x - M_{n-k})}{N_{n-k}}S_{n-k}(x) - \frac{S_{n-k+1}(x)}{N_{n-k}}\right]L_{n}^{\alpha}(x) = \\
D_{n-k-1}(x)L_{n-1}^{\alpha}(x) + S_{n-k-1}(x)L_{n}^{\alpha}(x).
\end{aligned}$$ ☑

Now we will prove a property about of the polynomials $D_{n-k}(x)$ and $S_{n-k}(x)$ that we will need later.

Theorem 6. *For each $k, n \in \mathbb{N}$ and $k \leq n$, $D_{n-k}(x)$ and $S_{n-k}(x)$ are polynomials with degree $k-1$ and $k-2$, respectively.*

Proof. If $k = 1$, $D_{n-1}(x) = 1$. So $\deg(1) = 0$. Assume that the property is valid for every $t \leq k$. We have that

$$D_{n-(k+1)}(x) = \frac{(x - M_{n-k})D_{n-k}(x)}{N_{n-k}} - \frac{D_{n-k+1}(x)}{1N_{n-k}}.$$

From the induction hypothesis, $\deg\big(D_{n-k}(x)\big) = k-1$ and $\deg\big(D_{n-k+1}(x)\big) = k-2$. Thus

$$\deg\big(D_{n-k-1}(x)\big) = 1 + \deg\big(D_{n-k}(x)\big) = k.$$

The proof for $S_{n-k}(x)$ is similar. ☑

Now, we proceed to write the Laguerre polynomials $L_n^{\alpha+j+1}(x)$ and $L_{n-1}^{\alpha+j+1}(x)$ as a combination of the polynomials $\big(\widetilde{L}_n^{\alpha}(x)\big)'$ and $\widetilde{L}_n^{\alpha}(x)$. According to (26) and (29), we have

$$\begin{aligned}
\widetilde{L}_n^{\alpha}(x) &= \sum_{k=0}^{j+1} A_n^{[k]} L_{n-k}^{\alpha+j+1}(x) \\
&= \sum_{k=0}^{j+1} A_n^{[k]} \Big[D_{n-k}(x) L_{n-1}^{\alpha+j+1}(x) + S_{n-k,x} L_n^{\alpha+j+1}(x)\Big] \\
&= \left[\sum_{k=0}^{j+1} A_n^{[k]} D_{n-k}(x)\right] L_{n-1}^{\alpha+j+1}(x) + \left[\sum_{k=0}^{j+1} A_n^{[k]} S_{n-k}(x)\right] L_n^{\alpha+j+1}(x). \quad (30)
\end{aligned}$$

If we denote

$$h_j(x) = \left[\sum_{k=0}^{j+1} A_n^{[k]} D_{n-k}(x)\right] \quad \text{and} \quad f_{j-1}(x) = \left[\sum_{k=0}^{j+1} A_n^{[k]} S_{n-k}(x)\right], \qquad (31)$$

then we have

$$\widetilde{L}_n^{\alpha}(x) = f_{j-1}(x) L_n^{\alpha+j+1}(x) + h_j(x) L_{n-1}^{\alpha+j+1}(x). \qquad (32)$$

Taking derivatives with respect to x in (30), multiplying by x and applying (10), we get

$$\begin{aligned}
x\big(\widetilde{L}_n^{\alpha}(x)\big)' &= \sum_{k=0}^{j+1} A_n^{[k]} x \big(L_{n-k}^{\alpha+j+1}(x)\big)' \\
&= \sum_{k=0}^{j+1} A_n^{[k]} \Big[(n-k) L_{n-k}^{\alpha+j+1}(x) + (n-k)(n-k+\alpha+j+1) L_{n-k-1}^{\alpha+j+1}(x)\Big]
\end{aligned}$$

and, applying (29) on the previous equation, we obtain

$$x\big(\widetilde{L}_n^{\alpha}(x)\big)' = \sum_{k=0}^{j+1} A_n^{[k]}\Big[(n-k)\Big(D_{n-k}(x)L_{n-1}^{\alpha+j+1}(x) + S_{n-k}(x)L_n^{\alpha+j+1}(x)\Big)$$
$$+ (n-k)(n-k+\alpha+j+1)\Big(D_{n-k-1}(x)L_{n-1}^{\alpha+j+1}(x) + S_{n-k-1}(x)L_n^{\alpha+j+1}(x)\Big)\Big]$$
$$= \sum_{k=0}^{j+1} A_n^{[k]}\Big[(n-k)D_{n-k}(x)L_{n-1}^{\alpha+j+1}(x) +$$
$$(n-k)(n-k+\alpha+j+1)D_{n-k-1}(x)L_{n-1}^{\alpha+j+1}(x)\Big] +$$
$$\sum_{k=0}^{j+1} A_n^{[k]}\Big[(n-k)S_{n-k}(x)L_n^{\alpha+j+1}(x) +$$
$$(n-k)(n-k+\alpha+j+1)S_{n-k-1}(x)L_n^{\alpha+j+1}(x)\Big].$$

Thus,

$$x\big(\widetilde{L}_n^{\alpha}(x)\big)' =$$
$$\sum_{k=0}^{j+1}\Big[A_n^{[k]}(n-k)D_{n-k}(x) + A_n^{[k]}(n-k)(n-k+\alpha+j+1)D_{n-k-1}(x)\Big]L_{n-1}^{\alpha+j+1}(x)$$
$$+\sum_{k=0}^{j+1}\Big[A_n^{[k]}(n-k)S_{n-k}(x) + A_n^{[k]}(n-k)(n-k+\alpha+j+1)S_{n-k-1}(x)\Big]L_n^{\alpha+j+1}(x).$$

If we define

$$g_{j+1}(x) = \sum_{k=0}^{j+1} A_n^{[k]}\Big[(n-k)D_{n-k}(x) + (n-k)(n-k+\alpha+j+1)D_{n-k-1}(x)\Big]$$

and

$$q_j(x) = \sum_{k=0}^{j+1} A_n^{[k]}\Big[(n-k)S_{n-k}(x) + (n-k)(n-k+\alpha+j+1)S_{n-k-1}(x)\Big], \quad (33)$$

then

$$x\big(\widetilde{L}^{\alpha}(x)\big)' = q_j(x)L_n^{\alpha+j+1}(x) + g_{j+1}(x)L_{n-1}^{\alpha+j+1}(x). \quad (34)$$

From (32) and (34), we obtain the following system of equations:

$$\begin{cases} \widetilde{L}_n^{\alpha}(x) & = f_{j-1}(x)L_n^{\alpha+j+1}(x) + h_j(x)L_{n-1}^{\alpha+j+1}(x); \\ x\big(\widetilde{L}^{\alpha}(x)\big)' & = q_j(x)L_n^{\alpha+j+1}(x) + g_{j+1}(x)L_{n-1}^{\alpha+j+1}(x). \end{cases}$$

Solving it, we obtain

$$L_n^{\alpha+j+1}(x) = \frac{\widetilde{L}_n^{\alpha}(x)g_{j+1}(x) - x\big(\widetilde{L}_n^{\alpha}(x)\big)' h_j(x)}{f_{j-1}(x)g_{j+1}(x) - h_j(x)q_j(x)}, \tag{35}$$

$$L_{n-1}^{\alpha+j+1}(x) = \frac{x\big(\widetilde{L}_n^{\alpha}(x)\big)' f_{j-1}(x) - \widetilde{L}_n^{\alpha}(x)q_j(x)}{f_{j-1}(x)g_{j+1}(x) - h_j(x)q_j(x)} \tag{36}$$

and, from (28), we get

$$x\big(\widetilde{L}_n^{\alpha}(x)\big)'' + (\alpha+j+2-)\big(\widetilde{L}_n^{\alpha}(x)\big)' =$$

$$-\sum_{k=0}^{j+1} A_n^{[k]}(n-k)\Big[D_{n-k}(x)L_{n-1}^{\alpha+j+1}(x) + S_{n-k}(x)L_n^{\alpha+j+1}(x)\Big] =$$

$$-\left[\sum_{k=0}^{j+1} A_n^{[k]}(n-k)D_{n-k}(x)\right]L_{n-1}^{\alpha+j+1}(x) - \left[\sum_{k=0}^{j+1} A_n^{[k]}(n-k)S_{n-k}(x)\right]L_n^{\alpha+j+1}(x).$$

Applying (35) and (36) on the previous expression, we have

$$x\big(\widetilde{L}_n^{\alpha}(x)\big)'' + (\alpha+j+2-x)\big(\widetilde{L}_n^{\alpha}(x)\big)' =$$

$$-\left[\sum_{k=0}^{j+1} A_n^{[k]}(n-k)D_{n-k}(x)\right]\left(\frac{x\big(\widetilde{L}_n^{\alpha}(x)\big)' f_{j-1}(x) - \widetilde{L}_n^{\alpha}(x)q_j(x)}{f_{j-1}(x)g_{j+1}(x) - h_j(x)q_j(x)}\right)$$

$$-\left[\sum_{k=0}^{j+1} A_n^{[k]}(n-k)S_{n-k}(x)\right]\left(\frac{\widetilde{L}_n^{\alpha}(x)g_{j+1}(x) - x\big(\widetilde{L}_n^{\alpha}(x)\big)' h_j(x)}{f_{j-1}(x)g_{j+1}(x) - h_j(x)q_j(x)}\right).$$

Thus,

$$x\big(\widetilde{L}_n^{\alpha}(x)\big)'' + \Bigg[(\alpha+j+2-x) + \frac{xf_{j-1}(x)\sum_{k=0}^{j+1} A_n^{[k]}(n-k)D_{n-k}(x)}{f_{j-1}(x)g_{j+1}(x) - h_j(x)q_j(x)} -$$

$$\frac{xh_j(x)\sum_{k=0}^{j+1} A_n^{[k]}(n-k)S_{n-k}(x)}{f_{j-1}(x)g_{j+1}(x) - h_j(x)q_j(x)}\Bigg]\big(\widetilde{L}_n^{\alpha}(x)\big)' +$$

$$\Bigg[-\frac{q_j(x)\sum_{k=0}^{j+1} A_n^{[k]}(n-k)D_{n-k}(x)}{f_{j-1}(x)g_{j+1}(x) - h_j(x)q_j(x)} +$$

$$\frac{g_{j+1}(x)\sum_{k=0}^{j+1} A_n^{[k]}(n-k)S_{n-k}(x)}{f_{j-1}(x)g_{j+1}(x) - h_j(x)q_j(x)}\Bigg]\widetilde{L}_n^{\alpha}(x) = 0.$$

As a consequence, we have proved the following theorem.

Theorem 7. *Let* $\{L_n^{\alpha}(x)\}_{n=0}^{\infty}$ *be the Laguerre polynomials and* $\{\widetilde{L}_n^{\alpha}(x)\}_{n=0}^{\infty}$ *the orthogonal polynomials associated with the inner product* (2)*. If we consider the same notation as in* (29)*,* (31) *and* (33)*, then*

$$Q(x;n)\big(\widetilde{L}_n^{\alpha}(x)\big)'' + W(x;n)\big(\widetilde{L}_n^{\alpha}(x)\big)' + R(x;n)\widetilde{L}_n^{\alpha}(x) = 0, \tag{37}$$

where

$$Q(x;n) = x\big[f_{j-1}(x)g_{j+1}(x) - h_j(x)q_j(x)\big],$$

$$W(x;n) = \Big[(\alpha + j + 2 - x)\big[f_{j-1}(x)g_{j+1}(x) - h_j(x)q_j(x)\big] + \\ xf_{j-1}(x)\sum_{k=0}^{j+1} A_n^{[k]}(n-k)D_{n-k}(x) - xh_j(x)\sum_{k=0}^{j+1} A_n^{[k]}(n-k)S_{n-k}(x)\Big]$$

and

$$R(x;n) = \\ \left[-q_j(x)\sum_{k=0}^{j+1} A_n^{[k]}(n-k)D_{n-k}(x) + g_{j+1}(x)\sum_{k=0}^{j+1} A_n^{[k]}(n-k)S_{n-k}(x)\right]. \tag{38}$$

6. An Electrostatic Model

As an application of (37), we present an electrostatic model that responds to the zeros distribution of the Laguerre-Sobolev type orthogonal polynomials. For $n \in \mathbb{N}$, let $\{x_{n,k}\}_{1\leq k\leq n}$ be the zeros of $\widetilde{L}_n^{\alpha}(x)$. Thus, for every $k \in \{1,\dots,n\}$, from (37) we get

$$Q(x_{n,k},n)\big(\widetilde{L}_n^{\alpha}(x_{n,k})\big)'' + W(x_{n,k},n)\big(\widetilde{L}_n^{\alpha}(x_{n,k})\big)' = 0$$

or, equivalently,

$$\frac{\big(\widetilde{L}_n^{\alpha}(x_{n,k})\big)''}{\big(\widetilde{L}_n^{\alpha}(x_{n,k})\big)'} = -\frac{W(x_{n,k},n)}{Q(x_{n,k},n)}. \tag{39}$$

Therefore,

$$\frac{W(x_{n,k},n)}{Q(x_{n,k},n)} = \frac{(\alpha+j+2)}{x_{n,k}} - 1 + \frac{f_{j-1}(x_{n,k})\sum_{k=0}^{j+1} A_n^{[k]}(n-k)D_{n-k}(x_{n,k})}{f_{j-1}(x_{n,k})g_{j+1}(x_{n,k}) - f_j(x_{n,k})g_j(x_{n,k})} \\ - \frac{f_j(x_{n,k})\sum_{k=0}^{j+1} A_n^{[k]}(n-k)S_{n-k}(x_{n,k})}{f_{j-1}(x_{n,k})g_{j+1}(x_{n,k}) - f_j(x_{n,k})g_j(x_{n,k})}. \tag{40}$$

Since $F(x_{n,k}) := f_{j-1}(x_{n,k})g_{j+1}(x_{n,k}) - f_j(x_{n,k})g_j(x_{n,k})$ is a polynomial of degree at most $2j-2$, then there exists $t_1 \in \{1,2,\dots,2j-2\}$, such that $x_1^{(F)},\dots,x_{t_1}^{(F)}$ are all the zeros of $F(x)$ different from 0.

In the general case, if $F(x)$ has complex and real zeros, we consider $I_1 \subset \{0,1,\dots,t_1\}$ such that, for each $i \in I_1$, $x_i^{(F)}$ is a complex number (but not real) and, for each $i \in J_1$, $x_i^{(F)}$ is a real number, where

$$J_1 = \{0,1,\dots,t_1\} - I_1.$$

Since for each $i \in I_1$, the complex conjugate of $x_i^{(F)}$ is also a zero of $F(x)$, then the number of elements in I_1 is even, say $2\widehat{n}$, where $\widehat{n}$ is an integer smaller than or equal to $j-1$. If $i_1,\dots,i_{2\widehat{n}}$ denotes an order of I_1 such that, for every $k \in \{1,3,\dots,2\widehat{n}-1\}$, $x_{i_{k+1}}^{(F)}$ is complex conjugate of $x_{i_k}^{(F)}$ then we define $\ddot{I}_1 = \big\{i_k \in I_1 : (1 \le k \le 2\widehat{n}) \wedge (k \text{ odd})\big\}$. For each $i \in \ddot{I}_1$ and $\ell \in J_1$, there exist real constants A, A_i, B_i, C_i, $\dot{A}_\ell$, such that

$$\frac{\big(\widetilde{L}_n^\alpha(x_{n,k})\big)''}{\big(\widetilde{L}_n^\alpha(x_{n,k})\big)'} = \frac{A}{x_{n,k}} + \sum_{i\in\ddot{I}_1} \frac{A_i x + B_i}{(x_{n,k})^2 + C_i^2} + \sum_{\ell\in J_1} \frac{\dot{A}_\ell}{x_{n,k} - x_\ell^{(F)}} = 0$$

and, since (*see* [21]),

$$\frac{\big(\widetilde{L}_n^\alpha(x_{n,k})\big)''}{\big(\widetilde{L}_n^\alpha(x_{n,k})\big)'} = -2\sum_{j=1,\, j\neq k}^{n} \frac{1}{x_{n,j} - x_{n,k}}, \tag{41}$$

we obtain,

$$\sum_{j=1,\, j\neq k}^{n} \frac{1}{x_{n,j} - x_{n,k}} + \frac{1}{2}\frac{A}{x_{n,k}} + \frac{1}{2}\sum_{i\in\ddot{I}_1} \frac{A_i x + B_i}{(x_{n,k})^2 + C_i^2} + \frac{1}{2}\sum_{\ell\in J_1} \frac{\dot{A}_\ell}{x_{n,k} - x_\ell^{(F)}} = 0. \tag{42}$$

Therefore, we can make the following electrostatic interpretation of the zeros of $\widetilde{L}_n(x)$. If we have $n \ge 1$ electric particles located on the complex plane under the interaction of a logarithmic external field and we have

$$\varphi_1(x) = \frac{1}{2}A\ln|x| + \frac{1}{2}\sum_{\ell\in J_1} \dot{A}_\ell \ln\left|x - x_\ell^{(F)}\right| + \frac{1}{2}\sum_{i\in\ddot{I}_1}\left[A_i \ln\sqrt{x^2 + C_i^2} + \frac{B_i}{C_i}\tan^{-1}\left(\frac{x}{C_i}\right)\right],$$

then (42) indicates that the totally energy gradient

$$\varepsilon(X) = -\sum_{1\le j<i\le n} \ln|x_j - x_i| + \sum_{i=1}^{n} \varphi_1(x_i),$$

with $X=(x_1,\dots,x_n)$, vanishes at the points $(x_{n,1},\dots,x_{n,n})$. In other words, the set of the zeros of Laguerre-Sobolev type orthogonal polynomial $\widetilde{L}_n^\alpha(x)$ constitutes a critical point. However, it is an open problem to determine if this critical point is a maximum, a minimum, or neither.

For a more general study of the electrostatic interpretation of standard orthogonal polynomials, we refer the reader to [6], [7], [8] and [11].

7. Hypergeometric Representation

Our goal in this section is to express the polynomials $\widetilde{L}_n^\alpha(x)$ in terms of a hypergeometric function, as occurs with the Laguerre polynomials. We know from (26) that $\widetilde{L}_n^\alpha(x)=\sum_{k=0}^{j+1}A_n^{[k]}L_{n-k}^{\alpha+j+1}(x)$. So, taking into account (6), we have

$$\widetilde{L}_n^\alpha(x)=\sum_{k=0}^{j+1}A_n^{[k]}(-1)^{n-k}(\alpha+j+2)_{n-k}\sum_{t=0}^{\infty}\frac{(-n+k)_t}{(\alpha+j+2)_t}\frac{x^t}{t!}, \tag{43}$$

or, equivalently,

$$(-1)^n(\alpha+j+2)_n\times\sum_{t=0}^{\infty}\frac{(-n)_t}{(\alpha+j+2)_t}\frac{x^t}{t!}\times \left[\sum_{k=0}^{j+1}\frac{A_n^{[k]}(-1)^k(-n+t)(-n+t+1)\cdots(-n+k+t-1)}{(\alpha+j+2+n-k)\cdots(\alpha+j+2+n-1)(-n)\cdots(-n+k-1)}\right]. \tag{44}$$

If we define

$$\pi(t)=\sum_{k=0}^{j+1}\frac{A_n^{[k]}(-1)^k(-n+t)(-n+t+1)\cdots(-n+k+t-1)}{(\alpha+j+2+n-k)\cdots(\alpha+j+2+n-1)(-n)\cdots(-n+k-1)},$$

which is a polynomial such that $\deg\pi(t)=j+1$, then

$$\pi(t)=D_n(t+\xi_1)(t+\xi_2)\cdots(t+\xi_{j+1}),$$

where D_n is the leading coefficient of $\pi(t)$ and $\xi_1,\cdots,\xi_{j+1}$ are complex numbers. Therefore,

$$\begin{aligned}\widetilde{L}_n^\alpha(x)&=\\ &(-1)^n(\alpha+j+2)_nD_n\sum_{t=0}^{\infty}\frac{(-n)_t}{(\alpha+j+2)_t}\frac{x^t}{t!}(t+\xi_1)(t+\xi_2)\cdots(t+\xi_{j+1})=\\ &(-1)^n(\alpha+j+2)_nD_n\xi_1\cdots\xi_{j+1}\sum_{t=0}^{\infty}\frac{(-n)_t(1+\xi_1)_t\cdots(1+\xi_{j+1})_t}{(\alpha+j+2)_t(\xi_1)_t\cdots(\xi_{j+1})_t}\frac{x^t}{t!}.\end{aligned} \tag{45}$$

As a consequence,

Theorem 8. *For all* $n \in \mathbb{N}$,

$$\widetilde{L}_n^\alpha(x) = (-1)^n(\alpha+j+2)_n D_n(\xi_1\cdots\xi_{j+1})$$
$$\qquad {}_{j+2}F_{j+2}(-n, 1+\xi_1, \ldots, 1+\xi_{j+1}; \alpha+j+2, \xi_1, \ldots, \xi_{j+1}; x).$$

In other words, the polynomials $\widetilde{L}_n^\alpha(x)$ *can be expressed as a hypergeometric function of the form* ${}_{j+2}F_{j+2}$.

In [5], the reader can find the hypergeometric representation of the Laguerre-Sobolev type orthogonal polynomials for $j = 1$, which coincides with the previous formula.

8. Zeros

In order to study the behavior of the zeros of the Laguerre-Sobolev type orthogonal polynomials, we will need the next proposition. Its proof can be found in [1].

Proposition 9. *Let* $\{\widetilde{L}_n^\alpha(x)\}_{n=0}^\infty$ *be the sequence of monic orthogonal polynomials with respect to the inner product* (2). *Then, for all* $n \geqslant \overline{n}$, $\widetilde{L}_n^\alpha(x)$ *has* $n-\overline{n}$ *changes of sign in the support of the measure, where* $\overline{n}$ *is the number of masses in* (2).

We see that if $M_i > 0$ for $i = 1, \ldots, j$ then the previous proposition shows that $\widetilde{L}_n^\alpha(x)$ has at least $(n-j+1)$ zeros in the support of the measure. In the following theorem, we show that there exist at least $(n-j)$ zeros in the support of the measure. First, we present a lemma.

Lemma 10. *Let* $x_1, \ldots, x_k$ *be the different zeros with odd multiplicity of the polynomial* $\widetilde{L}_n^\alpha(x)$ *in* $(0,\infty)$. *Then, there exists* $\varphi(x)$ *such that* $\deg\varphi(x) = j$, *and if* $\rho(x) = (x-x_1)\cdots(x-x_k)$ *then* $(\varphi\rho)^{(i)}(0) = 0$ *for* $i = 1, \ldots, j$.

Proof.
Existence. Let $h(x) = \rho(x)\Big(\sum_{t=0}^j a_t x^t\Big)$ and such that $h(0) = 1$. Then, from the Leibniz rule, we have

$$h^{(i)}(x) = \sum_{k=0}^{i}\binom{i}{k}\left(\sum_{t=0}^{j} a_t x^t\right)^{(k)}\rho^{(i-k)}(x).$$

Then,

$$h^{(i)}(0) = \sum_{k=0}^{i}\frac{i!}{(i-k)!}a_k\rho^{(i-k)}(0) = 0 \qquad \text{for} \qquad 0 < i \leq j,$$

and thus we have

$$
\begin{aligned}
h^{(0)}(0) &= a_0\rho^{(0)}(0) = 1\\
h^{(1)}(0) &= \frac{1!}{1!}a_0\rho^{(1)}(0) + \frac{1!}{0!}a_1\rho^{(0)}(0) = 0\\
h^{(2)}(0) &= \frac{2!}{2!}a_0\rho^{(2)}(0) + \frac{2!}{1!}a_1\rho^{(1)}(0) + \frac{2!}{0!}a_2\rho^{(0)}(0) = 0\\
&\vdots\\
h^{(j)}(0) &= \frac{j!}{j!}a_0\rho^{(j)}(0) + \frac{j!}{(j-1)!}a_1\rho^{(j-1)}(0) + \cdots +\\
&\qquad\qquad \frac{j!}{1!}a_{j-1}\rho^{(1)}(0) + \frac{j!}{0!}a_j\rho^{(0)}(0) = 0
\end{aligned}
$$

or, equivalently, a linear system of equations of the form **Aa=y**,

$$
\begin{bmatrix}
\rho^{(0)}(0) & & & \\
\rho^{(1)}(0) & \rho^{(0)}(0) & & \\
\rho^{(2)}(0) & 2\rho^{(1)}(0) & 2\rho^{(0)}(0) & \\
\vdots & \vdots & \vdots & \\
\rho^{(j)}(0) & \frac{j!\rho^{(j-1)}(0)}{(j-1)!} & \cdots & j!\rho^{(0)}(0)
\end{bmatrix}
\begin{bmatrix} a_0\\ a_1\\ a_2\\ \vdots\\ a_j \end{bmatrix}
=
\begin{bmatrix} 1\\ 0\\ 0\\ \vdots\\ 0 \end{bmatrix}. \tag{46}
$$

And since $\det \mathbf{A} = \prod_{t=0}^{j} t!\rho^{(0)}(0) \neq 0$, then the polynomial coefficients are uniquely determined. Thus, we define $\varphi(x) = \sum_{t=0}^{j} a_t x^t$.

Degree. This is a particular case of Lemma 2.1 in [1].

Zeros of $\boldsymbol{\varphi(x)}$. Using the same argument, we can show that the zeros of $\varphi(x)$ are outside of $(0,\infty)$ ([1]). ☑

Theorem 11. *If $M_i > 0$, for $i = 0,\dots,j$, there exist $(n-j)$ zeros in the support of the measure.*

Proof. From the previous lemma, we know that $\varphi(x)$ has their zeros outside of the support of the measure. Then, $h(x)\widetilde{L}_n^{\alpha}(x) = r(x)\varphi(x)\rho^2(x)$, where $r(x)$ and $\varphi(x)$ do not change sign in $(0,\infty)$. Therefore, we have

$$
h(x)\widetilde{L}_n^{\alpha}(x) \geq 0 \quad \text{for} \quad x\in(0,\infty) \qquad \text{or} \qquad h(x)\widetilde{L}_n^{\alpha}(x) \leq 0 \quad \text{for} \quad x\in(0,\infty).
$$

Thus, by the continuity of the polynomial $h(x)\widetilde{L}_n^{\alpha}(x)$ and the intermediate-value theorem we have that

$$
h(0)\widetilde{L}_n^{\alpha}(0) = 0 \qquad \text{or} \qquad \operatorname{sgn}\left(\int_0^{\infty} h(x)\widetilde{L}_n^{\alpha}(x)\,d\mu\right) = \operatorname{sgn}\Big(h(0)\widetilde{L}_n^{\alpha}(0)\Big).
$$

As a consequence,

$$\left| \int_0^\infty h(x)\widetilde{L}_n^\alpha(x)xe^{-x}\,dx + h(0)\widetilde{L}_n^\alpha(0) \right| > 0$$

and, therefore, $\deg h(x) = j + k \geq \deg \widetilde{L}_n^\alpha(x) = n$. That is, $k \geq n - j$, showing that $\widetilde{L}_n^\alpha(x)$ has at least $n - j$ zeros in the support of the measure. ☑

9. Some Numerical Experiments about Zeros.

Using (26), we consider some examples of Laguerre-Sobolev type orthogonal polynomials in order to show the behavior of their zeros. First we analyze an example where for some fixed parameters, the Laguerre-Sobolev type polynomials remain unchanged if we change one of the masses.

Example 12. Let $M = M_0$, $M_1 = 1$, $n = 3$, $j = 1$ and $\alpha = 2$. Then, using MatLab software, we can find that $K_3^{(0,0)}(0,0) = 5$, $K^{(1,0)}(0,0) = K_3^{(0,1)}(0,0) = -2.5$, $K^{(1,1)}(0,0) = 1.5$, $L_3^{(0)}(0) = -60$ and $L_3^{(1)}(0) = 60$.

Then, from (17) we have

$$\begin{cases} (1+5M)\big(\widetilde{L}_3^2\big)^{(0)}(0) - 2.5\big(\widetilde{L}_3^2\big)^{(1)}(0) & = -60 \\ -2.5M\big(\widetilde{L}_3^2\big)^{(0)}(0) + (1+1.5)\big(\widetilde{L}_3^2\big)^{(1)}(0) & = 60 \end{cases},$$

and making the computations, the solutions are

$$\begin{cases} \big(\widetilde{L}_3^2\big)^{(1)}(0) = 24 \quad \text{and} \quad \big(\widetilde{L}_3^2\big)^{(0)}(0) = 0 \quad \text{if} \quad M \neq -\frac{2}{5} \\ \big(\widetilde{L}_3^2\big)^{(1)}(0) = -\frac{2}{5}\big(\widetilde{L}_3^2\big)^{(0)}(0) + 24 \quad \text{if} \quad M = -\frac{2}{5} \end{cases}.$$

Since $M_0 \geq 0$, then $\big(\widetilde{L}_3^2\big)^{(0)}(0) = 0$ and $\big(\widetilde{L}_3^2\big)^{(1)}(0) = 24$. Furthermore, we know from (24) and (26) that

$$c_{(k;i)} = \sum_{m=i}^{2}\sum_{t=0}^{2}\binom{2}{t}\frac{(-1)^{m+i}m!}{(m-t)!\Gamma(3+i)(m-i)!}\delta_{k,m-t},$$

and

$$A_n^{[k]} = -\sum_{i=0}^{1} c_{(3-k;i)}M_i\widetilde{P}_n^{(i)}(0) + \binom{2}{k}(3)_{-k}.$$

Therefore,

$$A_3^{[k]} = -24c_{(3-k;1)} + \binom{j+1}{k}(3)_{-k},$$

where

$$c_{(3-1;1)} = -\frac{(1)^3}{(1)!\Gamma(4)} = -\frac{1}{6} \qquad \text{and}$$

$$c_{(3-2;1)} = \binom{2}{0}\frac{(-1)^2 1!}{(1)!\Gamma(4)(0)!} + \binom{2}{1}\frac{(-1)^3 2!}{(1)!\Gamma(4)(1)!} = -\frac{1}{2}.$$

As a consequence,

$$A_3^{[0]} = 1, \quad A_3^{[1]} = -24\left(-\frac{1}{6}\right) + \frac{(2)(3!)}{(2)!} = 10 \qquad \text{and}$$

$$A_3^{[2]} = -24\left(-\frac{1}{2}\right) + \binom{2}{2}\frac{3!}{(1)!} = 18$$

and, from (11),

$$L_{3-k}^4(x) = \sum_{t=0}^{3-k} \frac{(-1)^{3-k+t}(3-k)!\Gamma(3-k+4+1)x^t}{\Gamma(3-k-t+1)\Gamma(4+t+1)t!},$$

that is,

$$\widetilde{L}_3^2(x) = L_3^4(x) + 10L_2^4(x) + 18L_1^4(x) = x^3 - 11x^2 + 24x.$$

In other words, the Laguerre-Sobolev type orthogonal polynomial $\widetilde{L}_3^2(x)$ does not depend on M_0.

Example 13. In order to see how the zeros of $\widetilde{L}_4^\alpha(x)$ behave when the values of the masses vary, we present some computations performed in MatLab. It is important to remark that the results of the numerical experiments illustrates the Proposition 8 and Theorem 3. For $n = 4$ and several choices of α, we developed some numerical experiments which allow us to conjecture that:

(1) If the mass M_0 increases and M_1 is fixed, the last zero (arranged in decreasing order) increases.

(2) If the mass M_1 increases and M_0 is fixed, the last zero (arranged in decreasing order) decreases.

Now, setting $j = 1$, $n = 4$, and fixing one of the masses, our goal is to determine the value of the other mass such that the polynomial $\widetilde{L}_n^\alpha(x)$ has a negative zero. We will do this for fixed values of $\alpha = 3$ and $\alpha = -0.5$.

The following tables show the variation of the last zero of $\widetilde{L}_4^\alpha(x)$ when one of the masses is fixed and the other one varies. The approximated "critical" value of the varying mass (the value that causes a negative zero) is shown on the caption of the corresponding table.

M_0	$\widetilde{L}_4^{\alpha}(x) \qquad M_1 = 1 \qquad \alpha = 3$	Zeros
0.1	$x^4 - 22.7704x^3 + 147.5038x^2 - 255.6271x - 34.5442$	12.8667; 6.9666; 3.0629; -0.1258
0.5	$x^4 - 22.8096x^3 + 148.4729x^2 - 262.6052x - 20.2004$	12.8648; 6.9636; 3.0551; -0.0738
1	$x^4 - 22.8285x^3 + 148.9393x^2 - 265.9631x - 13.2982$	12.8638; 6.9621; 3.0512; -0.0487
10	$x^4 - 22.8598x^3 + 149.7122x^2 - 271.5277x - 1.8598$	12.8623; 6.9596; 3.0447; -0.0068
100	$x^4 - 22.8643x^3 + 149.8248x^2 - 272.3382x - 0.1937$	12.8621; 6.9593; 3.0437; -0.0007
1000	$x^4 - 22.8648x^3 + 149.8365x^2 - 272.4230x - 0.0195$	12.8620; 6.9592; 3.0436; -0.0001
10^4	$x^4 - 22.8649x^3 + 149.8377x^2 - 272.4315x$	12.8620; 6.9592; 3.0436; 0

TABLE 1. $M_1 = 1$; $M_0 \approx 10^4$.

M_1	$\widetilde{L}_4^{\alpha}(x) \qquad M_0 = 1 \qquad \alpha = 3$	Zeros
0.1	$x^4 - 24.3286x^3 + 175.7100x^2 - 388.7389x + 103.0158$	13.2109; 7.3279; 3.4844; 0.3054
0.5	$x^4 - 23.5200x^3 + 161.2800x^2 - 322.5600x + 40.3200$	13.0123; 7.1213; 3.2526; 0.1338
0.8574	$x^4 - 22.9997x^3 + 151.9942x^2 - 279.9734x$	12.8989; 6.9999; 3.1009; 0
1	$x^4 - 22.8285x^3 + 148.9393x^2 - 265.9631x - 13.2982$	12.8638; 6.9621; 3.0512; −0.0487
10	$x^4 - 20.3604x^3 + 104.8934x^2 - 63.9594x - 204.6701$	12.4594; 6.5268; 2.4159; −1.0418
100	$x^4 - 19.6699x^3 + 92.5699x^2 - 7.4413x - 258.2135$	12.3734; 6.4375; 2.2805; −1.4215
1000	$x^4 - 19.5882x^3 + 91.1123x^2 - 0.7565x - 264.5465$	12.3639; 6.4277; 2.2658; −1.4692
10^4	$x^4 - 19.5799x^3 + 90.9639x^2 - 0.0758x - 265.1914$	12.3629; 6.4267; 2.2643; −1.4740

TABLE 2. $M_0 = 1$; $M_0 \approx 0.8574$.

M_0	$\widetilde{L}_4^{\alpha}(x)$ $\quad M_1 = 1 \quad \alpha = -0.5$	Zeros
0.1	$x^4 - 11.4932x^3 + 28.7232x^2 - 4.5111x - 4.0701$	7.9670; 3.2871; 0.5315; −0.2924
0.5	$x^4 - 11.4652x^3 + 28.5699x^2 - 4.7602x - 3.2078$	7.9556; 3.2737; 0.4882; −0.2523
1	$x^4 - 11.4434x^3 + 28.4505x^2 - 4.9542x - 2.5361$	7.9467; 3.2633; 0.4504; −0.2171
10	$x^4 - 11.3783x^3 + 28.0941x^2 - 5.5331x - 0.5318$	7.9206; 3.2325; 0.2955; −0.0703
100	$x^4 - 11.3630x^3 + 28.0102x^2 - 5.6695x - 0.0597$	7.9146; 3.2253; 0.2332; −0.0100
1000	$x^4 - 11.3613x^3 + 28.0006x^2 - 5.6850x - 0.0060$	7.9139; 3.2245; 0.2239; −0.0011
10^5	$x^4 - 11.3611x^3 + 27.9996x^2 - 5.6867x$	7.9138; 3.2244; 0.2229; 0

TABLE 3. $M_1 = 1$; $M_0 \approx 10^5$.

M_1	$\widetilde{L}_4^{\alpha}(x)$ $M_0 = 1$ $\alpha = -0.5$	Zeros
0.1	$x^4 - 12.5752x^3 + 38.9473x^2 - 24.9535x + 0.1805$	8.1928; 3.5196; 0.8555; 0.0073
0.1122	$x^4 - 12.5017x^3 + 38.2656x^2 - 23.6548x$	8.1746; 3.5004; 0.8264; 0
0.5	$x^4 - 11.6684x^3 + 30.5375x^2 - 8.9306x - 1.9960$	7.9904; 3.3078; 0.5164; −0.1462
1	$x^4 - 11.4434x^3 + 28.4505x^2 - 4.9542x - 2.5361$	7.9467; 3.2633; 0.4504; −0.2171
10	$x^4 - 11.1941x^3 + 26.1387x^2 - 0.5496x - 3.1344$	7.9010; 3.2176; 0.3909; −0.3154
100	$x^4 - 11.1661x^3 + 25.8794x^2 - 0.0556x - 3.2015$	7.8960; 3.2127; 0.3851; −0.3277
1000	$x^4 - 11.1633x^3 + 25.8532x^2 - 0.0056x - 3.2083$	7.8955; 3.2122; 0.3846; −0.3289
10^4	$x^4 - 11.1630x^3 + 25.8505x^2 - 0.0006x - 3.2090$	7.8955; 3.2121; 0.3845; −0.3291

TABLE 4. $M_0 = 1$; $M_0 \approx 0.1122$.

Acknowledgements

The authors thank the valuable comments from the referees. They greatly contributed to improve the presentation of the manuscript. The third and fourth authors were supported by Promep and Conacyt.

References

[1] M. Alfaro, G. López, and M. L. Rezola, *Some Properties of Zeros of Sobolev-Type Orthogonal Polynomials*, Journal of Computational and Applied Mathematics **69** (1996), 171–179.

[2] G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, Encyclopedia of Mathematics and its Applications, vol. 71, Cambridge University Press, Cambridge, UK, 1999.

[3] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, USA, 1978.

[4] H. Dueñas and F Marcellán, *The Laguerre-SoboleV-Type Orthogonal Polynomials*, Journal of Approximation Theory **162** (2010), 421–440.

[5] H. Dueñas and F. Marcellán, *The Laguerre-Sobolev-Type Orthogonal Polynomials. Holonomic Equation and Electrostatic Interpretation*, Rocky Mount. Jour. Math **41** (2011), no. 1, 95–131.

[6] F. Grünbaum, *Variations on a Theme of Heine and Stieltjes: An Electrostatic Interpretation of the Zeros of Certain Polynomials*, Journal of Comp. and App. Math. **99** (1998), 189–194.

[7] Mourad E. H. Ismail, *An Electrostatic Model for Zeros of General Orthogonal Polynomials*, Pacific J. Math **193** (1999), no. 2, 355–369.

[8] ______, *More on Electrostatics Model for Zeros Of Orthogonal Polynomials*, Pacific. Journal. Of Numer. Funct. Anal. and Optimiz. **21** (2000), 191–204.

[9] ______, *Classical and Quantum Orthogonal Polynomials in One Variable*, Encyclopedia of Mathematics and its Applications, vol. 98, Cambridge University Press, Cambridge, UK, 2005.

[10] R. Koekoek, *Generalization of the Classical Laguerre Polynomials and some Q-Analogues*, Doctoral dissertation, Techn. Univ. of Delft, Delft, Netherlands, 1990.

[11] F. Marcellán, A. Martínez-Finkelshtein, and P. Martínez-González, *Electrostatic Models for Zeros of Polynomials: Old, New, and some Open Problems*, Journal of Comp. and App. Math. **207** (2007), no. 2, 258–272.

[12] G. Szegö, *Orthogonal Polynomials*, Colloquium Publications - American Mathematical Society, vol. 23, American Mathematical Society, Providence, USA, 4th ed., 1975.

(Recibido en agosto de 2012. Aceptado en enero de 2013)

Departamento de Matemáticas
Universidad Nacional de Colombia
Facultad de Ciencias
Carrera 30, calle 45
Bogotá, Colombia
e-mail: haduenasr@unal.edu.co
e-mail: jcgarciaar@unal.edu.co

Facultad de Ciencias
Universidad de Colima
Bernal Díaz del Castillo No. 340
Colonia Villas San Sebastián C.P. 28045
Colima, Colima, México
e-mail: garzaleg@gmail.com
e-mail: grarceo@gmail.com