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A Simple Observation Concerning Contraction Mappings

Una simple observación acerca de las contracciones

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Abstract. In this short note we show that the results obtained by Walter in [4] remain valid if we change the metric σ by another metric. Furthermore, if we use the norm $|\bullet|_{T,\epsilon}$ given in [3], Theorem B in[4] remains valid.

Key words and phrases. Contraction, contraction principle, fixed point.

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Resumen. En esta breve nota se muestra que los resultados obtenidos por Walter en [4] siguen siendo válidos si se cambia la métrica σ por otra. Además, si se utiliza la norma $|\bullet|_{T,\epsilon}$ usada en [3], el Teorema B en [4] sigue siendo válido.

Palabras y frases clave. Contracción, principio de la contracción, punto fijo.

1. Introduction

The main motivation of this note was the paper by W. Walter [4]. Thus, we consider $(\mathbf{X}, \varrho)$ a metric space and $T : \mathbf{X} \to \mathbf{X}$ a nonlinear map. We say that T is Lipschitz continuous if there exists $\alpha \geqslant 0$ such that

$$\varrho(Tx, Ty) \leqslant \alpha\varrho(x, y), \qquad \forall x, y \in \mathbf{X},$$

and if in addition $0 \leqslant \alpha < 1$, the map T is called a contraction.

The aim of this short note is to prove the following propositions and make some remarks about them.

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Proposition 1. *Let* $(\mathbf{X}, \varrho)$ *be a metric space and* $T : \mathbf{X} \to \mathbf{X}$ *a map such that, for a fixed* $n \in \mathbb{N}$, T^n *satisfies*

$$\varrho\big(T^n x, T^n y\big) \leqslant \alpha^n \varrho(x, y) \quad \textit{for} \quad x, y \in \mathbf{X}. \tag{1}$$

Then the function ζ *defined by*

$$\zeta(x, y) := \left[\varrho^2(x, y) + \frac{1}{\alpha^2}\varrho^2(Tx, Ty) + \cdots + \frac{1}{\alpha^{2(n-1)}}\varrho^2\big(T^{n-1}x, T^{n-1}y\big)\right]^{1/2} \tag{2}$$

is a metric on $\mathbf{X}$, *and* T *satisfies*

$$\zeta(Tx, Ty) \leqslant \alpha\zeta(x, y) \quad \textit{for} \quad x, y \in \mathbf{X}. \tag{3}$$

Moreover, there exist positive constants a, b *such that*

$$a\varrho(x, y) \leqslant \zeta(x, y) \leqslant b\varrho(x, y) \tag{4}$$

if and only if T *is Lipschitz continuous with respect to* ϱ.

Proof. It is not difficult to see that ζ is a metric on $\mathbf{X}$ and $\varrho(x, y) \leqslant \zeta(x, y)$ for all $x, y \in \mathbf{X}$. Now, using the definition of ζ we get

$$\begin{aligned}
\zeta(Tx, Ty) &= \Bigg[\varrho^2(Tx, Ty) + \frac{1}{\alpha^2}\varrho^2\big(T(Tx), T(Ty)\big) + \cdots \\
&\qquad\qquad + \frac{1}{\alpha^{2(n-1)}}\varrho^2\big(T^{n-1}(Tx), T^{n-1}(Ty)\big)\Bigg]^{1/2} \\
&= \Bigg[\varrho^2(Tx, Ty) + \frac{1}{\alpha^2}\varrho^2(T^2x, T^2y) + \cdots + \frac{1}{\alpha^{2(n-2)}}\varrho^2\big(T^{n-1}x, T^{n-1}y\big) \\
&\qquad\qquad + \frac{1}{\alpha^{2(n-1)}}\varrho^2\big(T^n x, T^n y\big)\Bigg]^{1/2} \\
&\leqslant \Bigg[\varrho^2(Tx, Ty) + \frac{1}{\alpha^2}\varrho^2(T^2x, T^2y) + \cdots + \frac{1}{\alpha^{2(n-2)}}\varrho^2\big(T^{n-1}x, T^{n-1}y\big) \\
&\qquad\qquad + \frac{\alpha^{2n}}{\alpha^{2(n-1)}}\varrho^2(x, y)\Bigg]^{1/2} \\
&\leqslant \Bigg[\alpha^2\bigg(\varrho^2(x, y) + \frac{1}{\alpha^2}\varrho^2(Tx, Ty) + \cdots + \frac{1}{\alpha^{2(n-1)}}\varrho^2\big(T^{n-1}x, T^{n-1}y\big)\bigg)\Bigg]^{1/2} \\
&\qquad\qquad = \alpha\zeta(x, y), \quad \forall x, y \in \mathbf{X},
\end{aligned}$$

where in the last inequality we have used (1). Hence (3) is proved.

Also, if $\zeta(x, y) \leqslant b\varrho(x, y)$, it is not difficult to show that T is Lipschitz continuous with respect to ϱ. In fact,

$$\varrho(Tx, Ty) \leqslant \zeta(Tx, Ty) \leqslant \alpha\zeta(x, y) \leqslant \alpha b\varrho(x, y), \quad \text{for all} \quad x, y \in \mathbf{X}.$$

Conversely, if T is Lipschitz continuous, then the powers of T are also Lipschitz continuous.

If we assume that

$$\varrho(T^k x, T^k y) \leqslant a_k \varrho(x,y), \quad x,y \in \mathbf{X}, \quad k = 1,2,\dots,n-1, \tag{5}$$

then

$$\varrho(x,y) \leqslant \zeta(x,y) \leqslant b\varrho(x,y), \quad \text{for} \quad x,y \in \mathbf{X} \tag{6}$$

where $b = 1 + a_1\alpha^{-1} + \cdots + a_{n-1}\alpha^{1-n}$. To get the last inequality we use the right side of (2) and (5). ☑

Proposition 2. *Let $(\mathbf{X}, |\bullet|)$ be a Banach space and $A \in \mathcal{L}(X)$ such that $|A^m| = \alpha^m$. Then the formula*

$$\|x\|_\zeta := \left(|x|^2 + \frac{1}{\alpha^2}|Ax|^2 + \cdots + \frac{1}{\alpha^{2(n-1)}}|A^{n-1}x|^2\right)^{1/2}$$

defines a norm on $\mathbf{X}$ equivalent to the original norm, and for the norm of A, $\|A\|_\zeta$, we have the inequality $\|A\|_\zeta \leqslant \alpha$.

Proof. It is not difficult to see that $\|\bullet\|_\zeta$ is a norm on $\mathbf{X}$ and $|x| \leqslant \|x\|_\zeta \leqslant b|x|$ for all $x \in \mathbf{X}$, i.e., the norms $|\bullet|$ and $\|\bullet\|_\zeta$ are equivalent. On the other hand,

$$\begin{aligned}
\|Ax\|_\zeta &= \left(|Ax|^2 + \frac{1}{\alpha^2}|A^2x|^2 + \cdots + \frac{1}{\alpha^{2(n-1)}}|A^n x|^2\right)^{1/2} \\
&\leqslant \left(|Ax|^2 + \frac{1}{\alpha^2}|A^2x|^2 + \cdots + \frac{1}{\alpha^{2(n-2)}}|A^{n-1}x|^2 + \frac{\alpha^{2n}}{\alpha^{2(n-1)}}|x|^2\right)^{1/2} \\
&\leqslant \left(\alpha^2\left[|x| + \frac{1}{\alpha^2}|A^2x|^2 + \cdots + \frac{1}{\alpha^{2(n-1)}}|A^{n-1}x|^2\right]\right)^{1/2} \\
&= \alpha\|x\|_\zeta.
\end{aligned}$$

This proves that $\|A\|_\zeta \leqslant \alpha$. ☑

2. Some Remarks

***Remark* 3.** Proposition 1 is the same as Proposition A in [4], where we change the metric σ by the metric ζ. Also, we can see that

$$\zeta(x,y) \leqslant \sigma(x,y) \quad \text{for all} \quad x,y \in \mathbf{X}. \tag{7}$$

The same applications given in [4] such as Contraction principle, Continuous dependence and Approximate iteration can also be obtained changing the metric σ by ζ. As an example, it is well known that if $(\mathbf{X}, \varrho)$ is a complete metric space and $T : \mathbf{X} \to \mathbf{X}$ is a contraction then there exists an unique $x \in \mathbf{X}$

such that $Tx = x$. This is called the *contraction principle* or the *Banach fixed point theorem.* For details on contraction principle see [1, p.120]. One way to find the fixed point x is: given $x_0 \in X$ arbitrary, the sequence $\{x_n\} \subset X$ given by

$$\begin{cases} x_0 \in X, \\ x_n = T^n x_0, \quad n = 0, 1, 2, \dots \end{cases} \tag{8}$$

converges to x. The recursion formula given in (8) is known as the *sucessive approximations method* to find the fixed point x. Moreover, we have *a priori error estimate*

$$\varrho(x_n, x) \leqslant \frac{\alpha^n}{1-\alpha}\varrho(x_0, x_1), \quad n = 0, 1, 2, \dots, \tag{9}$$

and *a posteriori error estimate*

$$\varrho(x_{n+1}, x) \leqslant \frac{\alpha}{1-\alpha}\varrho(x_n, x_{n+1}), \quad n = 0, 1, 2, \dots, \tag{10}$$

and, we have the *rate of convergence*

$$\varrho(x_{n+1}, x) \leqslant \alpha\varrho(x_n, x), \quad n = 0, 1, 2, \dots. \tag{11}$$

Now, if T is a map such that, for some $n \in \mathbb{N}$, T^n is a contraction with constant $\alpha^n < 1$ and T satisfies the hypothesis of Proposition 1 then from (3), we have that T is a contraction with respect to ζ with constant α. Thus, the inequalities (9), (10) and (11) remain valid if we change the metric ϱ by the metric ζ.

For numerical implementation it is important to know the number of iterations, N, to get a good approximation of the fixed point. Setting $d = \varrho(x, Tx)$ and using the a priori error estimate (9), we have a lower bound for N given by

$$N > \frac{\ln(\epsilon) + \ln(1-\alpha) - \ln d}{\ln K},$$

thus we have $\varrho(x_n, x) < \epsilon$, $\epsilon > 0$. For more details see [2].

***Remark* 4.** Proposition 2 is the same as Proposition B in [4] where we change the norm $\|\bullet\|$ by the norm $\|\bullet\|_\zeta$. Also, we can easily see that

$$\|x\|_\zeta \leqslant \|x\| \quad \text{for all} \quad x \in \mathbf{X}.$$

***Remark* 5.** The norm $\|\bullet\|_\zeta$ is the same norm $|\bullet|_{T,\epsilon}$ given in [3, p. 132]. If we use the norm $\|\bullet\|$ given in [4] which is equivalent to the norm $\|\bullet\|_\zeta$, the main result (Theorem 1) in [3] is still valid.

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