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An Alternative Proof of Hill's Criterion of Freeness for Abelian Groups

Una prueba alternativa del criterio de Hill para grupos abelianos libres

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Abstract. In this note we provide a different proof of Hill's criteria of freeness for abelian groups. Our proof hinges on the construction of suitable $G(\aleph_0)$-families of subgroups of the links in Hill's theorem and, ultimately, on the construction of such a family of pure subgroups of the group itself.

Key words and phrases. Abelian group, Freeness, Hill's criterion, $G(\aleph_0)$-family, Purity.

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Resumen. En este trabajo se proporciona una nueva demostración del criterio de Hill para grupos abelianos libres. La demostración se basa en la construcción de una $G(\aleph_0)$-familia de subgrupos en los eslabones del teorema de Hill y, prioritariamente, en la construcción de una familia tal de subgrupos puros.

Palabras y frases clave. Grupo abeliano, libertad, criterio de Hill, $G(\aleph_0)$-familia, pureza.

1. Introduction

In 1934, Lev Pontryagin proved that a countable, torsion-free abelian group is free if and only if every finite rank, pure subgroup is free [3]. Equivalently, every properly ascending chain of subgroups of the same finite rank is finite. From the proof of this criterion, it follows that a torsion-free abelian group G is free if there exists an ascending chain

$$0 = G_0 < G_1 < \cdots < G_n < \cdots , \qquad (n < \omega), \tag{1}$$

consisting of pure subgroups of G whose union is equal to G, such that every G_n is free and countable. Here, a subgroup H of the abelian group G is *pure* if solubility in G of every equation of the form $nx = h \in H$, with $n \in \mathbb{Z}$, implies its solubility in H. Also, we say that G is *torsion-free* if $n = 0$ or $g = 0$, whenever $n \in \mathbb{Z}$ and $g \in G$ satisfy $ng = 0$.

Later, in 1970, Hill established that, in order for an abelian group G to be free, it is sufficient to prove that it is the union of a countable ascending chain (1) consisting of free, pure subgroups [1]. In other words, he proved the following theorem, establishing thus that the countability condition on the cardinality of the links of the chain was superfluous.

Theorem 1 (Hill's criterion of freeness). *A torsion-free abelian group G is free if there exists a countable ascending chain*

$$0 = G_0 < G_1 < \cdots < G_n < \cdots , \qquad (n < \omega) \tag{2}$$

of subgroups of G, such that:

a) every G_n is free,

b) every G_n is a pure subgroup of G, and

c) $G = \bigcup_{n<\omega} G_n$.

In this note, we give a proof of Hill's criterion different from the one provided in [1]. Our proof hinges on the construction of suitable classes of subgroups of the groups G_n and, ultimately, on the construction of such a family consisting of pure subgroups of G. Section 3 of this work contains the proof of Theorem 1, while Section 2 presents some preliminary results.

2. Preparatory Lemmas

The following is a general result which will be used in the proof of Theorem 1. We refer to [2] for definitions of the set-theoretical concepts.

Lemma 1. *An abelian group G is free if there exists a continuous, well-ordered, ascending chain*

$$0 = A_0 < A_1 < \cdots < A_\gamma < A_{\gamma+1} < \cdots , \qquad (\gamma < \tau) \tag{3}$$

of subgroups of G, such that:

a) every factor group $A_{\gamma+1}/A_\gamma$ is free, and

b) $G = \bigcup_{\gamma<\tau} A_\gamma$.

Proof. The conclusion follows from the fact that G is isomorphic to the direct sum of the factor groups $A_{\gamma+1}/A_\gamma$, for $\gamma < \tau$. ☑

Recall that a $G(\aleph_0)$-*family* of an abelian group G is a collection $\mathcal{B}$ of subgroups of G, which satisfies the following properties:

i) 0 and G belong to $\mathcal{B}$,

ii) $\mathcal{B}$ is closed under unions of ascending chains, and

iii) for every $A_0 \in \mathcal{B}$ and every countable set $H \subseteq G$, there exists $A \in \mathcal{B}$ which contains both A_0 and H, such that A/A_0 is countable.

Clearly, every abelian group has a $G(\aleph_0)$-family, namely, the collection of all its subgroups.

For the rest of this section, we will assume the hypotheses of Theorem 1. Under these circumstances, we fix a basis X_n of G_n for every $n < \omega$, and let $\mathcal{B}_n$ be the family of all subgroups of G_n generated by subsets of X_n. Clearly, every member of G_n is a direct summand of G_n and, thus, a pure subgroup of G.

Lemma 2. *The collection $\mathcal{B}'_n = \{A \in \mathcal{B}_n \mid A + G_i$ is pure in G, for every $i < \omega\}$ is a $G(\aleph_0)$-family of pure subgroups of G_n, for every $n < \omega$.*

Proof. All we need to check is that the countability condition is satisfied, since the other conditions of a $G(\aleph_0)$-family are obvious. So, let $A_0 \in \mathcal{B}'_n$, and let H_0 be a countable subset of G_n. Moreover, let $m < \omega$, and assume that we have already constructed a chain

$$A_0 < A_1 < \cdots < A_m \tag{4}$$

of groups in $\mathcal{B}_n$, such that:

a) H_0 is contained in A_1,

b) for every $j < m$, the group A_{j+1}/A_j is countable, and

c) for every $j < m$ and every $i < \omega$, $(A_{j+1} + G_i)/(A_0 + G_i)$ contains the purification of $(A_j + G_i)/(A_0 + G_i)$ in $G/(A_0 + G_i)$.

To find the next member of (4), for every $i < \omega$, let $V_i \subseteq G_n$ be a complete set of representatives of the purification of $(A_m+G_i)/(A_0+G_i)$ in $G/(A_0+G_i)$. The sets V_i are clearly countable, so that $H_{m+1} = H_0 \cup \bigcup_{i<\omega} V_i$ is likewise countable. Therefore, there exists $A_{m+1} \in \mathcal{B}_n$ containing both A_m and H_{m+1}, such that A_{m+1}/A_m is countable. Inductively, we construct a chain

$$A_0 < A_1 < \cdots < A_m < \cdots, \qquad (m < \omega) \tag{5}$$

of groups in $\mathcal{B}_n$, satisfying properties a), b) and c) above, for every $m < \omega$.

Evidently, the union A of the links of (5) is a member of $\mathcal{B}_n$, A/A_0 is countable, and our construction guarantees that $(A+G_i)/(A_0+G_i)$ is pure in $G/(A_0+G_i)$. Thus, $A+G_i$ is pure in G and, consequently, A belongs to $\mathcal{B}'_n$. ☑

Lemma 3. *The collection* $\mathcal{B}=\{A<G \mid A\cap G_n\in\mathcal{B}'_n$*, for every* $n<\omega\}$ *is a* $G(\aleph_0)$*-family of pure subgroups of* G.

Proof. Again, only the countability condition merits attention; so, let $A_0\in\mathcal{B}$, and let $H\subseteq G$ be countable. For every $k<\omega$, let $A_k^0=A_0\cap G_k$. Moreover, let $n<\omega$, and assume that we have already constructed a finite ascending chain

$$A_0<A_1<\cdots<A_n \tag{6}$$

of subgroups of G, such that all factor groups A_m/A_0 are countable, for every $m\le n$. Furthermore, suppose that each link A_m in (6) may be expressed as the union of a countable ascending chain

$$0=A_0^m<A_1^m<\cdots<A_k^m<\cdots, \qquad (k<\omega) \tag{7}$$

of subgroups of G, such that:

a) $A_k^m\in\mathcal{B}'_k$, for every $k<\omega$ and every $m\le n$,

b) A_k^m is countable over $A_0\cap G_k$, for every $k<\omega$ and every $m\le n$, and

c) $A_k^m<A_m\cap G_k<A_k^{m+1}$, for every $k<\omega$ and $m+1\le n$.

For every $k<\omega$, the group $(A_n\cap G_k)/(A_0\cap G_k)$ is countable, so we may fix a countable set of representatives Y_k of $A_n\cap G_k$ modulo $A_0\cap G_k$. Moreover, there exists $B_k\in\mathcal{B}'_k$ containing both $A_0\cap G_k$ and Y_k, such that B_k is countable over $A_0\cap G_k$. Thus, any set of representatives H_k of B_k modulo $A_0\cap G_k$ is countable.

In order to construct the next link in (6), assume that the groups in the ascending chain $0=A_0^{n+1}<A_1^{n+1}<\cdots<A_k^{n+1}$ have been built as needed, for some $k<\omega$, and let $Z_k\subseteq G_k$ be a set of representatives of A_k^{m+1} modulo $A_0\cap G_k$. Then, there exists $A_{k+1}^{n+1}\in\mathcal{B}'_{k+1}$ which contains $A_0\cap G_{k+1}$ and the countable set $Z_k\cup H_{k+1}\cup(H\cap G_{k+1})$, such that A_{k+1}^{n+1} is countable over $A_0\cap G_{k+1}$.

Clearly, the group $A=\bigcup_{n<\omega}A_n$ contains both A_0 and H, and is countable over A_0. Moreover, our construction guarantees that $A\cap G_k\in\mathcal{B}_k$, for every $k<\omega$. We conclude that $A\in\mathcal{B}$. ☑

Before we prove our next result, it is important to notice that $A+G_n$ is a pure subgroup of G, for every $A\in\mathcal{B}$ and every $n<\omega$. Indeed, that

$(A+G_n)\cap G_{n+1}$ is pure in G follows from the fact that $A\cap G_{n+1}\in \mathcal{B}'_{n+1}$. Next, assume that $(A+G_n)\cap G_k$ is pure in G, for some $k>n$. It is easy to check that

$$\frac{(A+G_k)\cap G_{k+1}}{(A+G_n)\cap G_{k+1}} \cong \frac{G_k}{(A+G_n)\cap G_k}, \tag{8}$$

whence it follows that $(A+G_n)\cap G_{k+1}$ is pure in G. The claim is readily established after noticing that $A+G_n=\bigcup_{k<\omega}(A+G_n)\cap G_k$.

Lemma 4. *For every $A\in\mathcal{B}$, finite rank, pure subgroups of G/A are free.*

Proof. Let $A\in\mathcal{B}$, and let D be a pure subgroup of G containing A, such that D/A is of finite rank. If $S=\{d_1,\dots,d_n\}$ is a complete set of representatives of a maximal independent system of D modulo A, then there exists $k<\omega$ such that $S\subseteq G_k$. Then $A+(D\cap G_k)=D\cap(A+G_k)$ is a pure subgroup of G containing S, which lies between A and D. Therefore, $D=A+(D\cap G_k)$. The fact that $A\cap G_k\in\mathcal{B}'_k$ implies that $A\cap G_k$ is a summand of G_k. Therefore, there exists a finite rank, free group B, such that $D\cap G_k=(A\cap G_k)\oplus B$. Notice that

$$D=A+(D\cap G_k)=A+\big((A\cap G_k)\oplus B\big)=A\oplus B, \tag{9}$$

which implies that D/A is free. ☑

3. Proof of the Main Result

Proof of Theorem 1. Let α be any nonzero ordinal, and let

$$0=A_0<A_1<\cdots<A_\gamma<A_{\gamma+1}\cdots, \qquad (\gamma<\alpha) \tag{10}$$

be an ascending chain of subgroups in $\mathcal{B}$, such that all factor groups $A_{\gamma+1}/A_\gamma$ are free. If α is a limit ordinal, then we let $A_\alpha=\bigcup_{\gamma<\alpha}A_\gamma$. Otherwise, there exists an ordinal β such that $\alpha=\beta+1$. In this case, if there exists $x\in G\setminus A_\beta$, we let $A_{\beta+1}\in\mathcal{B}$ contain both x and A_β, such that $A_{\beta+1}/A_\beta$ be countable. Lemma 4 implies now that finite rank, pure subgroups of $A_{\beta+1}/A_\beta$ are free. Consequently, $A_{\beta+1}/A_\beta$ is free by Pontryagin's criterion.

Using transfinite induction, we construct a continuous, well-ordered, ascending chain (3) of subgroups of G satisfying properties (a) and (b) of Lemma 1. We conclude that G is free. ☑

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