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A Variational Characterization of the Fucik Spectrum and Applications

Una caracterización variacional del espectro de Fucik y aplicaciones

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Dedicated to Professor Alan C. Lazer, our inspiring teacher.

Abstract. We characterize the *Fucik spectrum* (see [9]) of a class selfadjoint operators. Our characterization relies on Lyapunov-Schmidt reduction arguments. We use this characterization to establish the existence of solutions for a semilinear wave equation. This work has been motivated by the authors' results in [4] where one dimensional second order ordinary differential equations are studied.

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Resumen. Se caracteriza el espectro de Fucik (véase [9]) de una clase de operadores autoadjuntos. Basamos esta caracterización en el método de reducción de Lyapunov-Schmidt. Usamos esta caracterización para demostrar la existencia de soluciones a una ecuación de onda semilineal. Este trabajo ha sido motivado por los resultados de los autores en [4] donde se estudian ecuaciones diferenciales ordinarias de segundo orden.

Palabras y frases clave. Espectro de Fucik, principio de puntos de silla, comportamiento asintótico.

1. Introduction

Let Ω be a measurable subset in $\mathbb{R}^n$ and L a selfadjoint operator with discrete spectrum acting on $L^2(\Omega)$, the space of square integrable functions in Ω. Examples of such operators are the Laplacian (Δ) subject to Dirichlet or

Neumann boundary conditions in smooth bounded regions, and the wave operator ($\Box \equiv \partial_{tt} - \partial_{xx}$) acting on 2π-periodic functions in the variable t that also satisfy the Dirichlet boundary condition $u(0,t) = u(\pi,t) = 0$ (see [2]).

The Fucik spectrum of L, $\mathcal{F}$, is the set of pairs $(a,b) \in \mathbb{R}^2$ for which the equation

$$Lu = au_+ - bu_- \qquad \text{in} \qquad \Omega \tag{1}$$

has a non-zero solution, where $u_+(x) = \max\{u(x), 0\}$, and $u_-(x) = \max\{-u(x), 0\}$. This concept was introduced by S. Fucik in [9] in the context of differential equations.

***Remark* 1.** If $u \neq 0$ satisfies (1) then $v = -u$ satisfies $Lv = bv_+ - av_-$. That is, $\mathcal{F}$ is symmetric with respect to the main diagonal in $\mathbb{R}^2$. Since $-L$ also has discrete spectrum, without loss of generality, we restrict our analysis to the case $b > a$. Also by adding to L an adequate multiple of the identity one may assume $b > a > 0$.

In order to establish our main result (Theorem 2 below) we recall the following global reduction principle (see [3]).

Theorem 1. *Let H be a separable real Hilbert space. Let X, Y be closed subspaces such that $H = X \oplus Y$, and $J : H \to \mathbb{R}$ a functional of class C^1. If there exists $m > 0$ such that*

$$\langle \nabla J(x_1 + y) - \nabla J(x_2 + y), x_1 - x_2 \rangle \leq -m \|x_1 - x_2\|^2 \tag{2}$$

for all $x_1, x_2 \in X$, $y \in Y$, then there exists a continuous function $r : Y \to X$ such that

- $J(y + r(y)) = \max\{J(y + x) \mid x \in X\}$.
- *$\widetilde{J} : Y \to \mathbb{R}$ defined by $\widetilde{J}(y) = J(y + r(y))$ is of class C^1.*
- *$x + y$ is a critical point of J if and only if $x = r(y)$ and y is critical point of $\widetilde{J}$.*

We let $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ and $0 \geq \lambda_0 > \lambda_{-1} > \cdots > \lambda_{-n} > \cdots$ denote the eigenvalues of L, and we assume that they do not have accumulation points in $\mathbb{R}$. That is, if the set $\{\lambda_i \mid i = 1, \ldots\}$ has infinitely many elements then $\lim_{i\to\infty} \lambda_i = +\infty$. Similarly, if the set $\{\lambda_{-i} \mid i = 1, \ldots\}$ has infinitely many elements then $\lim_{i\to\infty} \lambda_{-i} = -\infty$.

Let $\{\varphi_{j,k} \mid k = 1, 2, \ldots\}$ denote an orthonornal set of functions that span the set of eigenvectors corresponding to the eigenvalue λ_j. We will denote by $N(j)$ the multiplicity of the eigenvalue λ_j, which need not be finite. *We assume the set $\{\phi_{j,k} \mid j = 0, \pm 1, \ldots; k = 1, \ldots, N(j)\}$ to be complete in* $L^2(\Omega)$. Let H denote the subspace of $L^2(\Omega)$ of elements of the form

$$u = \sum_{j=-\infty,k=1}^{\infty,N(j)} a_{j,k}\varphi_{j,k} \tag{3}$$

such that

$$\sum_{j=-\infty,k=1}^{\infty,N(j)} |\lambda_j|(a_{j,k})^2 < \infty. \tag{4}$$

It is easily seen that H is a Hilbert space under the inner product

$$\left\langle \sum_{j=-\infty,k=1}^{\infty,N(j)} a_{j,k}\varphi_{j,k}, \sum_{j=-\infty,k=1}^{\infty,N(j)} b_{j,k}\varphi_{j,k} \right\rangle_1 = \sum_{j=-\infty,k=1}^{\infty,N(j)} (1+|\lambda_j|)a_{j,k}b_{j,k}. \tag{5}$$

We denote by $\|\cdot\|_1$ the norm defined by the inner product $\langle\ ,\ \rangle_1$.

We let $g_{a,b} \equiv g : \mathbb{R} \to \mathbb{R}$ be given by

$$g(t) = at \quad \text{for} \quad t \geq 0 \qquad \text{and} \qquad g(t) = bt \quad \text{for} \quad t \leq 0. \tag{6}$$

For u as in (3) and $v = \sum_{j=-\infty,k=1}^{\infty,N(j)} b_{j,k}\varphi_{j,k}$ we define

$$B(u,v) = \sum_{j=-\infty,k=1}^{\infty,N(j)} \lambda_j a_{j,k} b_{j,k}. \tag{7}$$

With u as in (3), let $J : H \to \mathbb{R}$ be defined by

$$J_{a,b}(u) \equiv J(u) = (1/2)\Big(B(u,u) - \int_\Omega u(x)g\big(u(x)\big)\,dx\Big). \tag{8}$$

Note that if $L(u) \in L^2(\Omega)$, i.e. if $\sum_{j=-\infty,k=1}^{\infty,N(j)} |\lambda_j^2|(a_{j,k})^2 < \infty$, then

$$B(u,v) = \big\langle L(u), u\big\rangle_0, \tag{9}$$

where $\langle\ ,\ \rangle_0$ denotes the usual inner product in $L^2(\Omega)$. Standard calculations prove that, for u as in (3) and $v = \sum_{j=-\infty,k=1}^{\infty,N(j)} b_{j,k}\varphi_{j,k}$,

$$\begin{aligned}\big\langle \nabla J(u), v\big\rangle_1 &= \lim_{t\to 0} \frac{J(u+tv)-J(u)}{t} \\ &= \sum_{j=-\infty,k=1}^{\infty,N(j)} \lambda_j a_{j,k} b_{j,k} - \int_\Omega g\big(u(x)\big)v(x)\,dx \qquad (10) \\ &= B(u,v) - \int_\Omega g\big(u(x)\big)v(x)\,dx.\end{aligned}$$

For $a \in (\lambda_j, \lambda_{j+1})$ and $b \geq a$, let X denote the closure of the subspace of H generated by the eigenfunctions corresponding to the eigenvalues λ_l with $l \leq j$, and Y the closure of the subspace generated by the eigenfunctions generated by the eigenvalues λ_l with $l > j$. Hence, for $x_1, x_2 \in X$ and $y \in Y$, we have

$$\begin{aligned}\big\langle \nabla J(x_1+y) - \nabla J(x_2+y), x_1 - x_2\big\rangle_1 \\ = B(x_1-x_2, x_1-x_2) - \int_\Omega (x_1-x_2)\big(g(x_1+y)-g(x_2+y)\big)\,d\xi \\ \leq B(x_1-x_2,x_1-x_2) - a\|x_1-x_2\|_0^2 \leq -m\|x_1-x_2\|_1^2, \quad (11)\end{aligned}$$

where $m \equiv m(a) = \inf\big\{(a-\lambda_i)/(1+|\lambda_i|) \mid i \leq j\big\} > 0$. Note that $m > 0$ since $\big\{(a-\lambda_i)/(1+|\lambda_i|)\big\}_i$ is either finite set of positive numbers or a sequence of positive numbers that converges to $+1$. Therefore (2) is satisfied and, hence, for each pair (a,b) there exists a continuous function $r_{a,b} \equiv r$ satisfying the properties in Theorem 1. For future reference, and using that g is homogeneous of degree one, we note that for any $x \in X$ and $\lambda > 0$ we have

$$\begin{aligned}0 &= \lambda\bigg(B(r(y),x) - \int_\Omega xg\big(y+r(y)\big)\,d\zeta\bigg) \\ &= B\big(\lambda r(y),x\big) - \int_\Omega xg\big(\lambda y + \lambda r(y)\big)\,d\zeta.\end{aligned} \qquad (12)$$

Hence

$$r(\lambda y) = \lambda r(y) \qquad \text{for any} \qquad \lambda > 0. \qquad (13)$$

In the next two lemmas we prove that the functions $r_{a,b}$ are compact and depend continuously on (a,b).

Lemma 1. *Let $N(l) < \infty$ for all $l > j$. If $\{y_n\}_n$ converges weakly to $\overline{y}$ then $\big\{r_{a,b}(y_n)\big\}_n$ contains a subsequence that converges to $r_{a,b}(\overline{y})$.*

Proof. For the sake of simplicity in the notation, throughout this proof we write r for $r_{a,b}$, and g for $g_{a,b}$. Let $\{y_n\}_n$ converge weakly to $\overline{y}$. Since

$$\begin{aligned}m\|r(y_n)\|_1^2 &\leq -\big\langle \nabla J_{a,b}\big(y_n + r(y_n)\big) - \nabla J_{a,b}(y_n), r(y_n)\big\rangle_1 \\ &= \big\langle \nabla J_{a,b}(y_n), r(y_n)\big\rangle_1 \\ &= -\int_\Omega g(y_n) r(y_n)\,d\xi \\ &\leq b\|y_n\|_0 \|r(y_n)\|_0,\end{aligned} \qquad (14)$$

the sequence $\big\{r(y_n)\big\}$ is bounded. Since $N(l) < \infty$ for all $l > j$, the imbedding of Y into $L^2(\Omega)$ is compact. Thus, without loss of generality, we may assume that $\{y_n\}$ converges in $L^2(\Omega)$ to $\overline{y}$. From the definition of r we have

$$\begin{aligned}(a-\lambda_j)\|r(y_n)-r(y_m)\|_0^2 + a\|y_n-y_m\|_0^2 \\ \leq -B\big(r(y_n)-r(y_m), r(y_n)-r(y_m)\big) \\ + \int_\Omega \big(g(y_n+r(y_n)) - g(y_m+r(y_m))\big)\big(y_n+r(y_n)-r(y_m)-y_m\big)\, d\zeta \\ = \int_\Omega \big(g(y_n+r(y_n)) - g(y_m+r(y_m))\big)(y_n-y_m)\, d\zeta. \quad (15)\end{aligned}$$

Since $\{y_n\}$ is a Cauchy sequence in $L^2(\Omega)$ and $\big\{g\big(y_n+r(y_n)\big)\big\}$ is bounded in $L^2(\Omega)$, the last term in (15) tends to zero, which proves that $\{r(y_n)\}$ is a Cauchy sequence in $L^2(\Omega)$. Let z be the limit of $\{r(y_n)\}$ in $L^2(\Omega)$. Hence $g(y_n+r(y_n))$ converges to $g(\overline{y}+z)$, and

$$0 = B(z,x) - \int_\Omega g(\overline{y}+z)x\, d\xi \quad (16)$$

for any $x \in X$. By the uniqueness of $r(\overline{y})$ we conclude that $z = r(\overline{y})$, which proves the lemma. ☑

Lemma 2. *If $\{(a_n,b_n)\}_n$ converges to (a,b), $b > a$, $b_n > a_n$ and $a, a_n \in (\lambda_j, \lambda_{j+1})$, then $\big\{r_{a_n,b_n}(y)\big\}_n$ converges to $r_{a,b}(y)$ for each $y \in Y$, i.e., r depends continuously on (a,b).*

Proof. Letting $z = r_{a_n,b_n}(y) - r_{a,b}(y)$, from the definition of r we have

$$\begin{aligned}0 &= B(z,z) - \int_\Omega \big(g_{a_n,b_n}\big(y+r_{a_n,b_n}(y)\big) - g_{a,b}\big(y+r_{a,b}(y)\big)\big)z\, d\xi \\ &= B(z,z) - \int_\Omega \big(g_{a_n,b_n}\big(y+r_{a_n,b_n}(y)\big) - g_{a_n,b_n}\big(y+r_{a,b}(y)\big)\big)z\, d\xi \\ &\quad - \int_\Omega \big(g_{a_n,b_n}\big(y+r_{a,b}(y)\big) - g_{a,b}\big(y+r_{a,b}(y)\big)\big)z\, d\xi. \quad (17)\end{aligned}$$

From (11), (17), and the fact that $(g_{a_n,b_n}(t) - g_{ab}(t))/t$ converges to 0 uniformly for $t \in \mathbb{R}$ as $n \to \infty$, we have

$$m\|z\|_1^2 \leq \big\|g_{a_n,b_n}\big(y+r_{a,b}(y)\big) - g_{a,b}\big(y+r_{a,b}(y)\big)\big\|_0 \|z\|_0. \quad (18)$$

Hence, given $\epsilon > 0$ there exists N such that if $n \geq N$ then

$$m\|z\|_1 \leq \big\|g_{a_n,b_n}\big(y+r_{a,b}(y)\big) - g_{a,b}\big(y+r_{a,b}(y)\big)\big\|_0 \leq \epsilon, \quad (19)$$

which proves the lemma. ☑

Our main result is the following.

Theorem 2. *If* $a \in (\lambda_j, \lambda_{j+1})$*,* $N(l) < \infty$ *for* $l \geq j+1$*, and* $b_1(a) \equiv b_1 = \sup\big\{b \geq a \mid \widetilde{J}_{a,\beta}(y) = J_{a,\beta}\big(y + r_{a,\beta}(y)\big) > 0$ *for all* $\beta \in (a,b), y \in Y - \{0\}\big\}$*, then*

a) (a, b_1) *is in the Fucik spectrum when* $b_1 < +\infty$*.*

b) If $b \in [a, b_1)$ *then* (a, b) *is not in the Fucik spectrum.*

c) For $b > a$*,* (a,b) *is in the Fucik spectrum if and only if the restriction of* $\widetilde{J}_{a,b}$ *to* $\{y \in Y \mid \|y\|_1 = 1\}$ *has a critical point on* $\{y \in Y \mid \|y\|_1 = 1, \widetilde{J}_{a,b} = 0\}$*.*

d) The function $b_1 : (\lambda_j, \lambda_{j+1}) \to [0, +\infty]$*,* $a \to b_1(a)$ *is non-increasing and continuous.*

***Remark* 2.** In general, even when X is finite dimensional, $b_1(a)$ need not be finite for all $a \in (\lambda_j, \lambda_{j+1})$. For example, it is easily seen that for $a \in (0, 0.25]$ the equation

$$-u'' = au_+ - bu_- \quad \text{in} \quad (0,\pi), \qquad u'(0) = u'(\pi) = 0 \tag{20}$$

has no non-trivial solution. That is, $b_1(a) = +\infty$ for all $a \in (0, 0.25]$. In this case $\lambda_0 = 0$ and $\lambda_1 = 1$.

In Lemma 7 we present a sufficient condition for $b_1(a)$ to be finite for all $a \in (\lambda_j, \lambda_{j+1})$. See Remark 3 for an application of Lemma 7.

For recent results on variational characterizations of the Fucik spectrum the reader is referred to [10] and [11] where a different variational characterization of the Fucik spectrum is provided. Unlike the results of [10] and [11], Theorem 2 includes operators L with infinitely many positive and infinitely many negative eigenvalues which may have infinite multiplicity. This allows for applications to non-elliptic problems such as the wave equation (21) below. Theorem 2 was motivated by the authors' work in [4] where the existence of periodic solutions for a semilinear ordinary differential equation is established using that the corresponding potential is asymptotically equal to $ug_{a,b}(u)/2$ with (a,b) not in the Fucik spectrum. For other results on the Fucik spectrum the reader is referred to [1, 6, 5, 8, 7, 12]; none of which study (1) in the generality presented here.

As an application of Theorem 2 we establish the existence of weak solutions for the semilinear wave equation

$$\begin{aligned} u_{tt}(x,t) - u_{xx}(x,t) &= h\big(u(x,t)\big) + p(x,t), && \text{for } x \in (0,\pi), t \in \mathbb{R} \\ u(x,t) &= u(x, t+2\pi), && \text{for } x \in (0,\pi), t \in \mathbb{R}, \\ u(0,t) &= u(\pi,t) = 0, && \text{for } t \in \mathbb{R}. \end{aligned} \tag{21}$$

where $h : \mathbb{R} \to \mathbb{R}$ is a continuous function, $p \in L^2\big((0,\pi) \times (0,2\pi)\big)$, and p is 2π-periodic in the variable t. The spectrum of $\Box = \partial_{tt} - \partial_{xx}$, D'Alembert's operator is given by $\{k^2 - j^2 \mid k = 1,2,\dots, j = 0,1,\dots\}$. Thus $\lambda_0 = 0$, $\lambda_1 = 1$. We assume that $h'(t) \geq \epsilon > 0$ for all $t \in \mathbb{R}$. We let $H(s) = \int_0^s h(t)\,dt$, and assume that that there exists positive real numbers a, b such that

$$\limsup_{s\to+\infty} \frac{2H(s)}{s^2} = a, \qquad \limsup_{s\to-\infty} \frac{2H(s)}{s^2} = b, \tag{22}$$

$$a \in (0,1) \quad \text{and} \quad b \in \big(a, b_1(a)\big), \tag{23}$$

where $b_1 \equiv b_1(a)$ is as in Theorem 2.

Using Theorem 2 we prove the following result.

Theorem 3. *If* (22) *and* (23) *hold, then the equation* (21) *has a weak solution.*

For the version of Theorem 3 to ordinary differential equations see [4]. The reader is invited to compare this result with Theorem 1 of [2] where an existence result for (21) is established when (a, b) is restricted to the rectangle $(0,1) \times (0,1)$.

2. Proof of Theorem 2

Without loss of generality we may assume that $a > 0$.

First we note that $b_1 \geq \lambda_{j+1}$. In fact, if $b \in [a, \lambda_{j+1})$ then, for $y \neq 0$,

$$\begin{aligned}
\widetilde{J}_{a,b}(y) &= J_{a,b}\big(y + r(y)\big) \\
&\geq J_{a,b}(y) \\
&= B(y,y) - \int_\Omega y(\xi) g_{a,b}\big(y(\xi)\big)\,d\xi \\
&\geq B(y,y) - b\int_\Omega y^2(\xi)\,d\xi \\
&\geq \frac{\lambda_{j+1} - b}{\lambda_{j+1}} B(y,y) \\
&> 0.
\end{aligned} \tag{24}$$

Next we relate the Fucik spectrum of L with the critical points of $J_{a,b}$.

Lemma 3. *The pair* $(a,b) \in \mathcal{F}$ *if and only if* $J_{a,b}$ *has a nonzero critical point.*

Proof. If $u \neq 0$ is a solution to (1) then multiplying (1) by v and using (9) we have

$$\begin{aligned} 0 &= \langle L(u), v\rangle_0 - \int_\Omega g_{a,b}(u)v\,d\zeta \\ &= B(u,v) - \int_\Omega g_{a,b}(u)v\,d\zeta \\ &= \langle \nabla J_{a,b}(u), v\rangle_1 . \end{aligned} \tag{25}$$

Thus u is a critical point of $J_{a,b}$.

On the other hand, if $u = \sum_{j=-\infty,k=1}^{\infty,N(j)} a_{j,k}\varphi_{j,k} \neq 0$ is a critical point of $J_{a,b}$ letting

$$u_{l-} = \sum_{j=-l,k=1}^{0,\min\{N(j),l\}} a_{j,k}\varphi_{j,k} \qquad \text{and} \qquad u_{l+} = \sum_{j=1,k=1}^{l,\min\{N(j),l\}} a_{j,k}\varphi_{j,k}, \tag{26}$$

we see that $L(u_{l-}), L(u_{l+}) \in H$ and $\{u_{l-} + u_{l+}\}_l$ converges to u in H, hence in $L^2(\Omega)$. Thus $0 = \langle \nabla J_{a,b}(u), L(u_{l+}) - L(u_{l-})\rangle_1$. This and the fact that $L(u_{l+})$ and $L(u_{l-})$ are in orthogonal subspaces give

$$\begin{aligned} \|L(u_{l+}) + L(u_{l-})\|_0^2 &= \|L(u_{l+}) - L(u_{l-})\|_0^2 \\ &= \sum_{j=-l,k=1}^{0,\min\{N(j),l\}} \lambda_{j,k}^2 a_{j,k}^2 + \sum_{j=1,k=1}^{l,\min\{N(j),l\}} \lambda_{j,k}^2 a_{j,k}^2 \\ &= B\big(u, L(u_{l+}) - L(u_{l-})\big) \\ &= \int_\Omega \big(L(u_{l+}) - L(u_{l-})\big) g_{a,b}(u) \\ &\leq \|L(u_{l+}) - L(u_{l-})\|_0 \|g_{a,b}(u)\|_0 . \end{aligned} \tag{27}$$

Thus $\big\{\|L(u_{l+}) + L(u_{l-})\|_0^2\big\}_l$ is bounded, which implies that $\big\{L(u_{l-} + u_{l+})\big\}_l$ defines a Cauchy sequence in $L^2(\Omega)$. Since L si assumed to be selfadjoint, hence closed, u is in the domain of L. That is $L(u) \in L^2(\Omega)$. Hence for all $v \in L^2(\Omega)$

$$\int_\Omega v g_{a,b}(u) = B(u,v) = \langle L(u), v\rangle_0 . \tag{28}$$

Thus $L(u) = g_{a,b}(u) = au_+ - bu_-$, which proves the lemma. ☑

Lemma 4. *If* $b \in [a, b_1)$ *then* $(a,b) \notin \mathcal{F}$.

Proof. By the definition of b_1, if $b \in [a, b_1)$ then $\widetilde{J}_{a,b}(y) > 0$ for any $y \in Y$ with $\|y\| = 1$. Hence

$$\begin{aligned}
&\big\langle \nabla J_{a,b}\big(y + r(y)\big), y + r(y)\big\rangle_1 \\
&\qquad = B\big(y + r(y), y + r(y)\big) - \int_\Omega \big(y + r(y)\big) g_{a,b}\big(y + r(y)\big)\, d\zeta \\
&\qquad = 2J_{a,b}\big(y + r(y)\big) \\
&\qquad = 2\widetilde{J}_{a,b}(y) \\
&\qquad > 0.
\end{aligned} \tag{29}$$

Thus, by Theorem 1, $\nabla J(y+x) \neq 0$ for $y+x \neq 0$, which proves the lemma. ☑

Lemma 5. *If $b_1(a) < \infty$ and $N(l) < \infty$ for all $l \geq j + 1$, then there exists $y_0 \in Y$ with $\|y_0\|_1 = 1$ and such that*

$$\widetilde{J}_{a,b_1}(y_0) = 0 = \min\big\{\widetilde{J}_{a,b_1}(y) \mid \|y\|_1 = 1\big\}.$$

Proof. By the definition of b_1 there exists a sequence $\{\beta_i\}_i$ converging to b_1 and a sequence $\{y_i\}_i$ with $\|y_i\|_1 = 1$ such that $\widetilde{J}_{a,\beta_i}(y_i) \leq 0$. Using again that $\lambda_j \to +\infty$ as $j \to \infty$, one sees that $\{y_i\}$ has a subsequence that converges strongly in $L^2(\Omega)$. For the sake of simplicity in the notations we denote by $\{y_i\}$ such a subsequence and denote by $\widehat{y}$ its weak limit in H which is its strong limit in $L^2(\Omega)$. Since, by the definition of X, Y, the functional J_{a,β_i} satisfies (2) we have

$$\begin{aligned}
m\|r_{a,\beta_i}(y_i)\|_1^2 &\leq -\big\langle \nabla J_{a,\beta_i}\big(y_i + r_{a,\beta_i}(y_i)\big) - \nabla J_{a,\beta_i}(y_i), r_{a,\beta_i}(y_i)\big\rangle_1 \\
&= \big\langle \nabla J_{a,\beta_i}(y_i), r_{a,\beta_i}(y_i)\big\rangle_1 \\
&= -\int_\Omega r_{a,\beta_i}(y_i) g_{a,\beta_i}(y_i)\, d\zeta.
\end{aligned} \tag{30}$$

Since $|g_{a,\beta_i}(t)| \leq c|t|$ for some constant c independent of i and t, we see that $\{r_{a,\beta_i}(y_i)\}$ is bounded in H. Let us also see that $\{r_{a,\beta_i}(y_i)\}_i$ is also a Cauchy sequence in H. In fact, letting $z_k = r_{a,b_k}(y_k)$ we have

$$\begin{aligned}
m\|z_i - z_j\|_1^2 &\leq -\big\langle \nabla J_{a,\beta_i}(y_i + z_i) - \nabla J_{a,\beta_i}(y_i + z_j), z_i - z_j\big\rangle_1 \\
&= B(z_j, z_i - z_j) - \int_\Omega (z_i - z_j)\big(g_{a,\beta_i}(y_i + z_j)\big)\, d\zeta \\
&= \int_\Omega (z_i - z_j)\big(g_{a,\beta_j}(y_j + z_j) - g_{a,\beta_i}(y_i + z_j)\big)\, d\zeta \\
&= \int_\Omega (z_i - z_j)\big(g_{a,\beta_j}(y_j + z_j) - g_{a,\beta_j}(y_i + z_j)\big)\, d\zeta \\
&\qquad + \int_\Omega (z_i - z_j)\big(g_{a,\beta_j}(y_i + z_j) - g_{a,\beta_i}(y_i + z_j)\big)\, d\zeta \\
&\equiv I_1 + I_2.
\end{aligned} \tag{31}$$

An elementary calculation shows that $|g_{a,\beta_j}(s) - g_{a,\beta_j}(t)| \leq \beta_j|s-t|$ for any $s,t \in \mathbb{R}$. Hence $\big\|\big(g_{a,\beta_j}(y_j+z_j) - g_{a,\beta_j}(y_i+z_j)\big)\big\|_0$ converges to 0 as i,j tend to infinity. This and the fact that $\{z_i\}_i$ is bounded in $L^2(\Omega)$ (see (30)) prove that the integral I_1 in (31) converges to zero as $i,j \to +\infty$. The term I_2 converges to zero as $i,j \to +\infty$ because $\{z_i\}_i$ is bounded in $L^2(\Omega)$ and $\{\beta_i\}_i$ converges. Let $\lim z_i = z \in X$. Therefore, for any $x \in X$, we have

$$\begin{aligned} 0 &= \lim_{i\to\infty}\left(B(z_i,x) - \int_\Omega x g_{a,\beta_i}(y_i+z_i)\,d\zeta\right) \\ &= B(z,x) - \int_\Omega x g_{a,b_1}(\widehat{y}+z)\,d\zeta, \end{aligned} \tag{32}$$

which implies that $z = r_{a,b_1}(\widehat{y})$.

From (30) we see that if $\widehat{y}=0$, $\lim_{i\to\infty}\|z_i\| = 0$. On the other hand, since $\widetilde{J}_{a,\beta_i}(y_i) \leq 0$ we have

$$\begin{aligned} 0 &\geq \limsup_{i\to\infty} 2\widetilde{J}_{a,\beta_i}(y_i) \\ &= \lim_{i\to\infty}\left(B(y_i,y_i) + B(z_i,z_i) - \int_\Omega (y_i+z_i) g_{a,\beta_i}(y_i+z_i)\,d\zeta\right), \end{aligned} \tag{33}$$

which contradicts that $B(y_i,y_i) \geq \big(\lambda_{j+1}/(\lambda_{j+1}+1)\big)\|y_i\|_1^2 = \lambda_{j+1}/(\lambda_{j+1}+1) > 0$ and $\lim_{i\to\infty}\big(B(z_i,z_i) - \int_\Omega (y_i+z_i)g_{a,\beta_i}(y_i+z_i)\,d\zeta\big) = 0$. Thus $\widehat{y} \neq 0$.

From the definition of r we have $0 = B(z_i,z_i) - \int_\Omega z_i g_{a,\beta_i}(y_i+z_i)\,d\zeta$. Thus

$$\begin{aligned} 2\widetilde{J}_{a,b_1}(\widehat{y}) &= B(\widehat{y},\widehat{y}) + B\big(r(\widehat{y}), r(\widehat{y})\big) - \int_\Omega \big(\widehat{y}+r(\widehat{y})\big) g_{a,b_1}\big(\widehat{y}+r(\widehat{y})\big)\,d\zeta \\ &\leq \liminf_{i\to\infty} B(y_i,y_i) - \int_\Omega \widehat{y} g_{a,b_1}\big(\widehat{y}+r(\widehat{y})\big)\,d\zeta \\ &= \liminf_{i\to\infty}\left(B(y_i,y_i) - \int_\Omega y_i g_{a,\beta_i}(y_i+z_i)\,d\zeta\right) \\ &\leq 0. \end{aligned} \tag{34}$$

Since $\widetilde{J}(\lambda y) = J\big(\lambda y + r(\lambda y)\big) = \lambda^2 J(y + r(y))$ we have $\widetilde{J}_{a,b_1}\big((1/\|\widehat{y}\|)\widehat{y}\big) \leq 0$, which proves that

$$\inf\big\{\widetilde{J}_{a,b_1}(y) \mid \|y\|_1 = 1\big\} \leq 0. \tag{35}$$

Assuming that $\widetilde{J}_{a,b_1}(y) < 0$ for some y with $\|y\|_1 = 1$, by the continuity of r for $\epsilon > 0$ close to zero we have $\widetilde{J}_{a,b_1-\epsilon}(y) < 0$. Since this contradicts the definition of b_1 we have $\inf\big\{\widetilde{J}_{a,b_1}(y) \mid \|y\|_1 = 1\big\} = 0$. Taking $y_0 = (1/\|\widehat{y}\|_1)\widehat{y}$ the lemma is proven. ☑

Lemma 6. *For y_0 as in Lemma 5 we have $\nabla\widetilde{J}(y_0) = 0$.*

Proof. Since y_0 is a critical point of $\widetilde{J}_{a,b_1}$ restricted to the unit sphere in H, by the Lagrange multipliers rule there exists $\lambda \in \mathbb{R}$ such that $\nabla \widetilde{J}_{a,b_1}(y_0) = \lambda y_0$. Thus

$$\begin{aligned} 0 &= 2\widetilde{J}_{a,b_1}(y_0) \\ &= B(y_0, y_0) + B\big(r(y_0), r(y_0)\big) - \int_\Omega \big(y_0 + r(y_0)\big) g_{a,b_1}\big(y_0 + r(y_0)\big)\, d\zeta \\ &= \big\langle \nabla \widetilde{J}_{a,b_1}(y_0), y_0 \big\rangle_1 \\ &= \lambda \langle y_0, y_0 \rangle_1, \end{aligned} \tag{36}$$

which implies that $\lambda = 0$ since $\|y_0\|_1 = 1$. Hence y_0 is a critical point of $\widetilde{J}_{a,b_1}$ which proves the lemma. ☑

Proof. (Theorem 2)

- Part a) of Theorem 2 follows from Lemmas 5-6.
- Part b) was proved in Lemma 4.
- Since also $\big\langle \nabla J_{a,b}(x+y), x+y \big\rangle = 2J(x+y) = \widetilde{J}(y)$ we have that the critical points of J are the critical points of $\widetilde{J}$ restricted to the unit sphere with $\widetilde{J}(y) = 0$, which proves part c).
- Now we prove part d). Let $\widehat{y}$ be such that

$$\begin{aligned} 0 &= \widetilde{J}_{a,b_1(a)}(\widehat{y}) = J_{a,b_1(a)}\big(\widehat{y} + r_{a,b_1(a)}(\widehat{y})\big) \\ &= \min\big\{ J_{a,b_1(a)}(y + r_{a,b_1(a)}(y)) \mid y \in Y, \|y\|_1 = 1 \big\}. \end{aligned} \tag{37}$$

  Since $L\big(\widehat{y} + r_{a,b_1(a)}(\widehat{y})\big) = g_{a,b_1(a)}\big(\widehat{y} + r_{a,b_1(a)}(\widehat{y})\big)$ and a is not an eigenvalue of L, $\widehat{y} + r_{a,b_1(a)}(\widehat{y})$ is not a positive function. Hence, letting $G_{a,b}(u) = (1/2)u g_{a,b}(u)$, for any $\delta > 0$ we have

$$\begin{aligned} &2\widetilde{J}_{a,b_1(a)+\delta}(\widehat{y}) \\ &= \max_{x \in X} \Big\{ B(x+\widehat{y}, x+\widehat{y}) - \int_\Omega G_{a,b_1(a)+\delta}(x+\widehat{y}) \Big\} \\ &= \max_{x \in X} \Big\{ B(x+\widehat{y}, x+\widehat{y}) - \int_\Omega G_{a,b_1(a)}(x+\widehat{y}) - \int_\Omega G_{0,\delta}(x+\widehat{y}) \Big\} \\ &= B\big(r_{a,b_1(a)+\delta}(\widehat{y}) + \widehat{y}, r_{a,b_1(a)+\delta}(\widehat{y}) + \widehat{y}\big) \\ &\qquad - \int_\Omega G_{a,b_1(a)}\big(r_{a,b_1(a)+\delta}(\widehat{y}) + \widehat{y}\big) - \int_\Omega G_{0,\delta}\big(r_{a,b_1(a)+\delta}(\widehat{y}) + \widehat{y}\big) \\ &< 0, \end{aligned} \tag{38}$$

where we have used that if $r_{a,b_1(a)+\delta}(\widehat{y}) \neq r_{a,b_1(a)}(\widehat{y})$, then

$$B\big(r_{a,b_1(a)+\delta}(\widehat{y}) + \widehat{y}, r_{a,b_1(a)+\delta}(\widehat{y}) + \widehat{y}\big) - \int_\Omega G_{a,b_1(a)}\big(r_{a,b_1(a)+\delta}(\widehat{y}) + \widehat{y}\big)\, d\zeta < 0, \quad (39)$$

while if $r_{a,b_1(a)+\delta}(\widehat{y}) = r_{a,b_1(a)}(\widehat{y})$ then $-\int_\Omega G_{0,\delta}\big(r_{a,b_1(a)+\delta}(\widehat{y}) + \widehat{y}\big)\, d\zeta < 0$ since $r_{a,b_1(a)}(\widehat{y}) + \widehat{y}$ is not a positive function.

Arguing as in (38) we see that for any $\delta \in (0, \lambda_{j+1} - a)$,

$$\widetilde{J}_{a+\delta,b_1(a)}(\widehat{y}) \leq 0. \quad (40)$$

Hence $b_1(a+\delta) \leq b_1(a)$, which proves that b_1 is a non-increasing function.

Let $\{a_n\}_n$ be a sequence in $(\lambda_j, \lambda_{j+1})$ converging to a. Suppose that $b_1(a_n) \leq b_1(a) - \delta$ for some $\delta > 0$. By the definition of $b_1(a_n)$ there exists $y_n \in Y$ with $\|y_n\|_1 = 1$ such that $\widetilde{J}_{a_n,b_1(a_n)}(y_n) = 0$. Since Y is compactly imbedded in $L^2(\Omega)$, we may assume without loss of generality that $\{y_n\}$ converges weakly to $\overline{y}$ in Y and that $\{y_n\}$ converges strongly to $\overline{y}$ in $L^2(\Omega)$. Since

$$B(y_n - y_m, y_n - y_m) = \int_\Omega (y_n - y_m)\big(g_n\big(y_n + r_n(y_n)\big) - g_m\big(y_m + r_m(y_m)\big)\big)\, d\zeta, \quad (41)$$

where $g_n = g_{a_n,b_1(a_n)}$, $r_n = r_{a_n,b_1(a_n)}$, similarly g_m, r_m. Hence $\{y_n\}_n$ converges strongly to $\overline{y}$ in H. Let $c \leq b_1(a) - \delta$ be a limit point of $\{b_1(a_n)\}_n$. Without loss of generality we may assume that $\{b_1(a_n)\}_n$ converges to c. Thus

$$\begin{aligned}\widetilde{J}_{a,c}(\overline{y}) &= J_{a,c}\big(\overline{y} + r_{a,c}(\overline{y})\big)\\ &= \lim_{n\to\infty} J_{a_n,b_1(a_n)}\big(\overline{y} + r_{a_n,b_1(a_n)}(\overline{y})\big)\\ &= \lim_{n\to\infty} J_{a_n,b_1(a_n)}\big(y_n + r_{a_n,b_1(a_n)}(y_n)\big)\\ &= 0,\end{aligned} \quad (42)$$

which contradicts the definition of $b_1(a)$. Hence

$$\liminf_{t\to a} b_1(t) \geq b_1(a). \quad (43)$$

From (38) we have

$$\begin{aligned}\limsup_{n\to\infty} \widetilde{J}_{a_n,b_1(a)+\delta}(\overline{y}) &= \limsup_{n\to\infty} J_{a_n,b_1(a)+\delta}\big(\overline{y} + r_{a_n,b_1(a)+\delta}(\overline{y})\big)\\ &= J_{a,b_1(a)+\delta}\big(\overline{y} + r_{a,b_1(a)+\delta}(\overline{y})\big)\\ &= \widetilde{J}_{a,b_1(a)+\delta}(\overline{y})\\ &< 0.\end{aligned} \quad (44)$$

Hence, for n sufficiently large, $b_1(a_n) \le b_1(a) + \delta$. Since $\delta > 0$ is arbitrary,

$$\limsup_{t \to a} b_1(t) \le b_1(a). \tag{45}$$

From (43) and (45) we conclude that b_1 is continuous, which concludes the proof of Theorem 2 ☑

3. A Sufficient Condition for $b_1(a) < \infty$

Lemma 7. *If $Y \setminus \{0\}$ contains a non-negative function then $b_1(a) < +\infty$ for all $a \in (\lambda_k, \lambda_{k+1})$.*

Proof. Let $y \in Y \setminus \{0\}$ be a non-negative function. Assuming that $\inf_{x \in X} \int_\Omega \big((-y+x)_-\big)^2 = 0$, there exists a sequence $\{x_k\} \in X$ such that

$$0 = \inf_{x \in X} \int_\Omega \big((-y+x)_-\big)^2 = \lim_{k \to \infty} \int_\Omega \big((-y+x_k)_-\big)^2. \tag{46}$$

Writing $2x_k = (-y+x_k) + (x_k+y) = (-y+x_k)_+ - (-y+x_k)_- + (y+x_k)$, and using (46) we have

$$\begin{aligned} 0 &= 2\int_\Omega x_k y \\ &= \lim_{k\to\infty} \int_\Omega \big((-y+x_k)_+ y + (y+x_k)y\big)\, d\zeta \\ &\ge \|y\|_0^2 \\ &> 0. \end{aligned} \tag{47}$$

This contradiction proves that $c = \inf_{x \in X} \int_\Omega \big((-y+x)_-\big)^2 > 0$. Now, for any $x \in X$,

$$\begin{aligned} 2J(-y+x) &= B(-y,-y) - a\|y\|_0^2 + B(x,x) - a\|x\|_0^2 \\ &\qquad - (b-a)\int_\Omega \big((-y+x)_-\big)^2\, d\xi \\ &\le B(y,y) - a\|y\|_0^2 - c(b-a) \\ &< 0, \end{aligned} \tag{48}$$

for $b > a + \big(B(y,y) - a\|y\|_0^2\big)/c$. Hence $\widetilde{J}(-y) = \max\{J(-y+x) \mid x \in X\} < 0$ and $b_1(a) \le a + \big(B(y,y) - a\|y\|_0^2\big)/c < +\infty$, which proves the lemma. ☑

4. Proof of Theorem 3

Let $W = (0,\pi) \times (0, 2\pi)$ and H be the vector space of elements $u \in L^2(W)$ with

$$u(x,t) = \sum_{k=1,j=0}^{\infty,\infty} a_{k,j} \sin(kx)\cos(jt) + b_{k,j}\sin(kx)\sin(jt) \tag{49}$$

and

$$\sum_{k=1,j=0}^{\infty,\infty} \big(1+|j^2-k^2|\big)(a_{k,j}^2+b_{k,j}^2)<\infty. \tag{50}$$

This vector space is a Hilbert space under the inner product defined by

$$\langle u,v\rangle_1=\sum_{k=1,j=0}^{\infty,\infty} \big(1+|j^2-k^2|\big)(a_{k,j}\alpha_{k,j}+b_{k,j}\beta_{k,j})\,\delta_{kj}, \tag{51}$$

where $\delta_{k0}=\pi^2$, $\delta_{kj}=\pi^2/2$ for $j>0$, u is as in (49), and v is given by

$$v(x,t)=\sum_{k=1,j=0}^{\infty,\infty} \alpha_{k,j}\sin(kx)\cos(jt)+\beta_{k,j}\sin(kx)\sin(jt). \tag{52}$$

For u,v as above, let

$$B(u,v)=\sum_{k=1,j=0}^{\infty,\infty} \delta_{kj}(k^2-j^2)(a_{k,j}\alpha_{k,j}+b_{k,j}\beta_{k,j}). \tag{53}$$

Note that if u is a function of class C^2 and $\Box u\in L^2(\Omega)$ then $B(u,v)=\langle\Box u,v\rangle_0$. Let

$$I(u)=\sum_{k=1,j=0}^{\infty,\infty} \frac{\delta_{kj}}{2}(k^2-j^2)\big(a_{k,j}^2+b_{k,j}^2\big)-\int_W(\Gamma(u)+pu)\,dx\,dt, \tag{54}$$

where $\Gamma(t)=\int_0^t h(s)\,ds$. We say that $u\in H$ is a *weak solution* to (21) if u is a critical point of I. Let X be the closure of the subspace of H generated by functions of the type $\sin(kx)\cos(jt),\sin(kx)\sin(jt)$ such that $k^2-j^2\le 0$, and Y the closure of the subspace of H generated by functions of the type $\sin(kx)\cos(jt)$, $\sin(kx)\sin(jt)$ such that $k^2-j^2\ge 1$. A straightforward calculation shows that

$$\langle\nabla I(u),v\rangle=B(u,v)-\int_W(h(u)+p)v\,dx\,dt. \tag{55}$$

Since $B(z,z)\le 0$ for any $z\in X$, for $y\in Y, z_1,z_2\in X$ we have

$$\begin{aligned}\big\langle\nabla I(y+z_1)-\nabla I(y+z_2),z_1-z_2\big\rangle=&\\ B(z_1-z_2,z_1-z_2)-\int_W\big(h(y+z_1)-h(y+z_2)\big)(z_1-z_2)\,dx\,dt&\\ \le -\epsilon\|z_1-z_2\|_1^2,&\end{aligned} \tag{56}$$

where $\|\cdot\|_1$ denotes the norm in H. Thus by Theorem 1 there exists a continuous function $\rho: Y\to X$ such that $u\in H$ is a critical point I if and only if

$u = y + \rho(y)$ with y a critical point of $\widetilde{I}(y) \equiv I\big(y + \rho(y)\big)$. By the continuity of the function b_1 (see Theorem 2) there exists $\delta > 0$ such that $a + \delta < 1$ and $b + \delta < b_1(a + \delta)$. By (22), there exists a real number C such that

$$\Gamma(t) \leq \frac{1}{2} t g_{a+\delta,b+\delta}(t) + C, \quad \text{for all} \quad t \in R. \tag{57}$$

For $x \in X$ and $y \in Y$, let

$$J_{a+\delta,b+\delta}(x + y) = \frac{1}{2}\bigg(B(x + y, x + y) - \int_W (x + y) g_{a+\delta,b+\delta}(x + y)\bigg) \tag{58}$$

Therefore, letting $w = r_{a+\delta,b+\delta}(y)$ we have

$$\begin{aligned}
\widetilde{I}(y) &= I\big(y + \rho(y)\big) \\
&\geq I(y + w) \\
&= \frac{1}{2} B(y + w, y + w) - \int_W \big(\Gamma(y + w) + p(x,t)(y + w)\big)\, dx\, dt \\
&\geq \frac{1}{2}\bigg(B(y + w, y + w) \\
&\qquad - \int_W \big(g_{a+\delta,b+\delta}(y + w) + p(x,t)\big)(y + w)\, dx\, dt - 2\pi^2 C\bigg) \\
&\geq \|y + w\|_1^2 \bigg(\frac{\widetilde{J}_{a+\delta,b+\delta}(y)}{\|y + w\|_1^2} - \frac{\|p\|_0}{\|y + w\|_1} - \frac{2\pi^2 C}{\|y + w\|_1^2}\bigg).
\end{aligned} \tag{59}$$

Let us see that $\inf\big\{\widetilde{J}_{a+\delta,b+\delta}(y) \mid \|y\| = 1\big\} \equiv A > 0$. Let $m = m(a + \delta) > 0$ be as in (11). Assuming that $\{y_k\}_k$ is a sequence in $\{y \in Y \mid \|y\|_1 = 1\}$ such that $\lim_{k\to\infty} \widetilde{J}(y_k) = 0$, by the compact imbedding of Y in $L^2(\Omega)$ we may assume that $\{y_k\}_k$ converges weakly in H and strongly in $L^2(\Omega)$. Let $\widehat{y}$ be such a limit and, for the sake of simplicity in the notations, let $J_{a+\delta,b+\delta} = J$, $r = r_{a+\delta,b+\delta}$, and $\widetilde{J}_{a+\delta,b+\delta} = \widetilde{J}$. Arguing as in (31) we see that $\{r(y_k)\}_k$ converges in H. Let $\widehat{x}$ be such a limit. Hence, for any $z \in X$,

$$\begin{aligned}
\big\langle J(\widehat{y} + \widehat{x}), z\big\rangle_1 &= B(\widehat{x}, z) - \int_W \big(g_{a+\delta,b+\delta}(\widehat{y} + \widehat{x})\big) z \\
&= \lim_{k\to\infty} B\big(r(y_k), z\big) - \int_W \big(g_{a+\delta,b+\delta}\big(y_k + r(y_k)\big)\big) z \\
&= 0.
\end{aligned} \tag{60}$$

Thus $\widehat{x} = r(\widehat{y})$ and

$$\begin{aligned}
2J(\widehat{x}+\widehat{y}) &= B(\widehat{x},\widehat{x}) + B\big(\widehat{y},\widehat{y}\big) - \int_W \big(g_{a+\delta,b+\delta}(\widehat{y}+\widehat{x})\big)(\widehat{y}+\widehat{x}) \\
&\le \liminf_{k\to\infty} B\big(r(y_k), r(y_k)\big) + B(y_k,y_k) \\
&\qquad\qquad - \int_W \big(g_{a+\delta,b+\delta}\big(y_k + r(y_k)\big)\big)\big(y_k + r(y_k)\big) \\
&= \liminf_{k\to\infty} \widetilde{J}(y_k) \\
&= 0.
\end{aligned} \tag{61}$$

Since $(a+\delta, b+\delta)$ is not in the Fucik spectrum of $\square$, we have $\widehat{x} = \widehat{y} = 0$. Thus $\lim_{k\to\infty} B\big(r(y_k), r(y_k)\big) - \int_W \big(g_{a+\delta,b+\delta}(y_k + r(y_k))\big)\big(y_k + r(y_k)\big) = 0$. On the other hand, from the definition of B (see (53)), $B(y_k, y_k) \ge \|y_k\|_1^2 = 1$, which contradicts that $\lim_{k\to\infty} \widetilde{J}(y_k) = 0$. Thus $A > 0$.

Now for $y \in Y$ and $\rho(y) = w \in X$,

$$\begin{aligned}
\widetilde{I}(y) &= \frac{1}{2}B(y+w,y+w) - \int_W \big(\Gamma(y+w) + p(x,t)(y+w)\big)\,dx\,dt \\
&\ge \frac{1}{2}\Bigg(B(y+w,y+w) \\
&\qquad - \int_W \big(g_{a+\delta,b+\delta}(y+w) + p(x,t)\big)(y+w)\,dx\,dt - 2\pi^2 C\Bigg) \\
&\ge \|y+w\|_1^2 \left(\frac{\widetilde{J}_{a+\delta,b+\delta}(y)}{\|y+w\|_1^2} - \frac{\|p\|_0}{\|y+w\|_1} - \frac{2\pi^2 C}{\|y+w\|_1^2}\right).
\end{aligned} \tag{62}$$

From (14) we see that there exists $c > 0$, independent of y such that $\|w\|_1 \le c\|y\|_1$. These and the fact that $\widetilde{J}$ is homogeneous of degree 2 (see (13)) yield

$$\begin{aligned}
\widetilde{I}(y) &\ge \|y+w\|_1^2\big(A\|y\|_1^2/\|y+w\|_1^2 - \|p\|_0/\|y+w\|_1 - 2\pi^2 C/\|y+w\|_1^2\big) \\
&\ge \|y+w\|_1^2\big(A/(1+c^2) - \|p\|_0/\|y+w\|_1 - 2\pi^2 C/\|y+w\|_1^2\big) \\
&\to +\infty \qquad as \qquad \|y\| \to +\infty.
\end{aligned} \tag{63}$$

Arguing as in Lemma 1 we see that

$$N(y) = \frac{1}{2}B\big(\rho(y),\rho(y)\big) - \int_\Omega \big(\Gamma\big(y+\rho(y)\big) + p\rho(y)\big)\,d\zeta \tag{64}$$

defines a weakly lower semicontinuous function. Thus $\widetilde{I}$ is the sum of a convex function ($y \to B(y,y)/2 - \int_\Omega pyd\zeta$) with a weakly lower semicontinuous function ($y \to N(y)$). Hence, by (63), $\widetilde{I}$ achieves its minimum at some point y_0. By Theorem 1 we conclude that $y_0 + \rho(y_0)$ is a critical point of I, hence a solutions to (21). This proves Theorem 3.

***Remark* 3.** Since $\sin(x) \in Y$, by Lemma 7, $b_1(a) < \infty$ for all $a \in (0, 1)$.

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