Lecturas Matemáticas
Volumen **29** *(2008), páginas 33–48*
ISSN 0120–1980

A systematization of fundamentals of multisets*

D. Singh
A. M. Ibrahim
Ahmadu Bello University, Nigeria
T. Yohanna
Kaduna State University, Nigeria
J. N. Singh†
Barry University, Miami FL, USA

Abstract. This paper presents a concise history of the uses of multisets in disguised forms which have eventually led to the formalization of multiset theory, and a systematization of representations of multisets and operations under multisets.

Key words and phrases. Multisets, history, representations, operations and orderings.

2000 AMS Mathematics Subject Classification. 03E15, 03E20, 03E30.

Resumen. Se presenta una historia concisa de los usos encubiertos de los multiconjuntos que han conducido finalmente a la formalización de la teoría de los multiconjuntos y a una sistematización de sus representaciones y a las operaciones entre ellos.

Introduction

Set theory was discovered (or invented) by Georg Ferdinand Ludwig Philip Cantor (1845–1918), a German Mathematician. In post-Sputnik era, besides numerous publications appearing in the core area of set theory, one can hardly come across a book in other areas of mathematics which does not begin with some discussion of set theory. In fact, set theory has eventually become the language of science.

---

*This paper was presented as a part of inaugural lecture organized by the Senate, Ahmadu Bello University, Zaria on 20th September 2006

†Corresponding author.

CANTOR's concept of a set:

> *"By a 'set', we are to understand any collection M of definite and distinct objects m of our intuition or thought (which will be called the 'elements' of M) into a whole."*

Accordingly, one of the underlying assumptions of Cantorian set theory dictates that no element shall be allowed to appear more than once. The collection $\{a, b, b, c\}$ consequently becomes a set only after deleting the repeated elements viz, $\{a, b, c\}$. Indeed, this aspect of Cantorian set theory did not go hand in hand with the requirements of various other sciences in seeking mathematical formulation of some of the challenging problems. For example, repeated roots of polynomial equations, repeated observations in statistical samples, repeated hydrogen atoms in a water molecule, H_2O, etc. need to be counted for attaining adequacy and exactness. The concept of multiset ensues once the repeated elements are admitted in a set.

A multiset (mset, for short) is an unordered collection of objects (called the elements) in which, unlike a standard (Cantorian) set, elements are allowed to repeat. In other words an mset is a set in which elements may belong more than once and hence it is a non-Cantorian set. KNUTH [Knu 81, p. 636] notes that "despite frequent occurrences of multisetlike structures in mathematics, there is currently no structured way to deal with multisets" and similar observations have been voiced by many others.

It is gratifying to note that during the recent years a sizeable number of papers, specially dealing with the development of the theory of multisets, have appeared and, not very surprisingly, some of the outcomes turned out to be exceedingly vexing. In part, it conforms to CANTOR's resolute insistence on not admitting repeated elements in a set.

One is tended to believe that time is getting ripe to have a clearer perspective of CANTOR's compassion for delimiting himself to the development of set theory in its simplest form, called the standard or crisp set theory. In our opinion, a convergent view point of multitude of researches that have gone into developing a comprehensive theory of multisets dictates that it should be CANTOR's fascinating theory of infinity that could not have straightforwardly got through in the generalized cases, such as multisets.

Over the years, besides sporadic evidences of applications of mulisets in Philosophy, Logic, Linguistics and Physics, a good number of them witnessed in mathematics and computer science which have led to the formulation of a comprehensive theory of multisets.

In this paper, we endeavour to present some of the history of multisets, a systematization of various approaches toward formalizing the concept of muliset and operations under multisets.

Some early history of multisets

The concept of multiple-membership collection is as old as the concept of number itself. For example, evidences of representing a number by a collection of tally marks or units are found in the work of the Babylonians (200 B.C.), Egyptians (3500 - 1700 B.C) and Greeks (see [Hal 84, p. 132] and [Ifr 85] for details). Knuth [Knu 81, p. 23] notes that enumeration of permutations of a set was known in ancient times and historically the first known document is the Hebrew *Book of Creation* (C. 100 A. D.), followed by the Indian Classic *Anuyogadvarā-sutra* (C. 500 A. D.); and the corresponding results for multiset seems to have appeared first in another Indian Classic *Lilávati* of Bháscara Achárya (C. 1150). He further notes that Kircher (C. 1650, pp. 5–7) correctly gave the number of permutations of multiset $\{m.C, n.D\}$ for several values of m and n, though without revealing his method of calculation except when $n = 1$. A generalization of the rule for enumerating the permutations of multiset appeared in Prestet's *Élémens de mathématiques* (Paris 1675, 351–352), and later in John Wallis' *Treatise of Algebra, 2* (Oxford 1685, pp. 117- 118). It is heartening to note that, by exploiting Dominique Floata's work done in 1965, Knuth [Knu 81, pp. 24–31], presents a good number of significant results on multiset permutations.

An early reference to multiset is also found in the work of Marius Nizolius (1498–1576), cf. [Ang 65] and [Sin 94]. "It is not perhaps clear whether Nizolius' multitude comes closer to 'class' or 'heap'. . . But Nizolius' multitudines, might still be heaps in the sense of Quine or Goodman" ([Ang 65, pp. 319–320]).

In [Bri 87, p. 1], it is noted with 'emphasis' (rightly, we think), which is missing in [Bli 91, p. 330]: ". . . attempting to 'make sense' of Boole's algebra of Logic that multisets do have a history, and... that honour should go to George Boole's [1854] *Laws of Thought*". Hailperin [Hai 86] has justified (by introducing signed heaps) that Boole's Laws of Thought may be interpreted as a treatise dealing with multisets. However, on the question whether Boole himself had this interpretation in mind, Brink [Bri 87, p. 2] notes that "Hailperin is silent on this point, but strong evidence comes from another quarter: at least one eminent 19th century Logician read Boole in precisely right way, and that was Charles Sanders Pierce". Brink in [Bri 78, p. 294] notes that "the axiom system for Boole's original system is an axiom system for signed heaps".

Cantor himself, in his first (1895) definition of a set, defines the cardinality M^* of a set M as a collection of 'units' (one unit for each element of M), admitting repetitions (for details, see [Hal 84, pp. 128–142] and [Can 55, p. 11])

Dedekind, in his 1888 masterpiece "Was sind und was sollen die Zahlen?" ("The nature and meaning of numbers", English translation, cf. [Bli 91, p. 80]) introduces the notion of msets. He considers a non-injective function ψ from the set Σ (with n elements) onto the set $\psi(\Sigma)$ (with m elements) to conclude that n is also the number of elements in $\psi(\Sigma)$ counted in this sense, while the number of its actually different elements is m. DEDEKIND remarks: "In this way, we reach the notion, very useful in many cases, of systems [sets] in which every element is endowed with a certain frequency number which indicates how often it is to be reckoned as element of the system". In fact, DEDEKIND's approach has been found to be the most fundamental.

WEIERSTRASS defines real numbers as certain msets of rational numbers in which finitely many repetitions are allowed. For example, the quantity $\pi = 3.141\ldots$ can be identified with a multiset containing the number 1 with multiplicity 3, the element $\frac{1}{10}$ with multiplicity 1, the element $\frac{1}{100}$ with multiplicity 4, etc. (see [Hal 84, p. 134]; [Bli 91, pp. 325–326]; and JOURDAN's observations in [Can 55, p. 16–18] for details).

[Whi 33] is indeed the first place where msets receive a substantive mathematical treatment, particularly in terms of algebra of characteristic functions of sets and subsequently, that of 'generalized' sets (sets whose characteristic functions are integer valued). He cites "chains in analysis situs" in which "each element is counted any number of times" [Whi 33, p. 412].

The appendix B, entitled *Ars combinatoria*, of [Wey 49] is another rich place where msets receive utmost vindication, in theoretical as well as applied directions. WEYL defines mset as a set with an equivalence relation defined on it. Equivalent elements are said to be "in the same state" ("in the same sort" in [Mon 87]). [Wey 49, pp. 238–239] notes: "... electrons may be in this or that position; atoms in a molecule may be N, He, Li, ... atoms.... no artificial differences between elements are introduced by their levels ... and merely the intrinsic differences of states are made use of ...". Wey1 applies these concepts to a variety of problems in physics, chemistry and genetics: for example, in physics, "Two individuals in the same 'complete state' (no further refinement is possible) are indiscernible by any intrinsic characters although they may not be the same thing" [Wey 49, p. 245]. [Wey 49, p. 238] notes: "... one has to distinguish between equal (= of the same kind) and identicals". Thus, an aggregate is identified by its distinct states which is precisely the concept of multiset.

RADO [Rad 75] defines an mset to be any cardinal-valued function whose nontrivial domain (the collection of elements not mapped to zero) is a set. The class of msets is called the cardinal module, like "a module over the semi-group of all cardinals" (p.135), An mset f represents a family of sets $\alpha = (x_i)_{i\in I}$ just in case $f(x_i)$ equals the number of times x_i occurs in α (that is, $f(x) = |\{i \in I : x_i = x\}|$ for all x). RADO also makes use of signed msets (functions which may have negative cardinal values).

Parker-Rhodes [Par 81, p. xiii] observes that "... there exists no branch of mathematics in which a third parity-relation, besides equality and inequality, is admitted". Parker-Rhodes [Par 81] develops a theory of "sorts" (collections of indistinguishables), an elaborate mathematical system, and applies it to explicate some fundamental problems of physics. "Copies of elements behave as identicals when they appear as elements of different classes, but as a plurality (each of them contributes to cardinality) when they are elements of the same class" (p. 7). Elements of multisets conform to the Parker-Rhodes principle of indistinguishables. However, the system of Parker-Rhodes is a radical departure from classical mathematics because of its triparitous nature (objects may be identical, distinct or twins). The underlying idea lies in treating equality and identity as different relations.

In course of formalizing category theory, Corcoran [Cor 80, 199–201] introduces msets, for example, $S_a[a,b]_{2,1} = [a,b]_{3,1}$, where S_a = the successor function of a.

Meyer and MacRobie [MM 82] make use of multisets in the study of relevant implication by way of reckoning how often a premise is repeated in the course of derivation.

The principle of indistinguishability of [Par 81] has been re-emphasized in [Wil 03]. Wildberger argues that two physical objects are either different or they are the same (or equal) but separate or they are coinciding (and identical). He takes the example of a water molecule with two hydrogen atoms, say H^1 and H^2 and one oxygen atom say O and concludes that H^1 and O are obviously different (and, for that matter, H^2 and O are different), however H^1 and H^2 are the same but separate, while H^1 and H^1 are coinciding and identical.

Wildberger [Wil 03, p. 3], by taking some examples from the real world, such as "...to a shopkeeper, any two dollar coins are equal even if their years of minting are different, where as a coin collector would regard them essentially different, etc.", proposes some very innovative conclusions like "the notion of equality is often a relative one", "traditional mathematics works with the implicit assumption that all objects exist uniquely in some ideal sense" and "relies heavily on equivalence relation", etc., which, for the time being, appear to us only a contrived idea.

Theory of multisets

Introduction. As to the development of a formal (axiomatic) theory of multisets, despite having voiced by a number of authors [Sin 94], [FBL 73], for example), their disagreement with Cantor's insistence on disallowing repeated elements in a set, nothing substantial came through for a considerable period of time. [Lak 76], inspired by Rado's remark that "there is no axiomatization for msets" during his 1974 London Mathematical Society lecture entitled " Multisets and multicardinals" , is known to be the first axiomatization of multiset theory. In [Lak 76], function (like VON NEUMANN's axiomatization) is taken as primitive. Lake admits: '... It might be though desirable to have an axiomatization which does not go via functions, such an axiomatization ... could be conveniently written out using $x \in_z y$ (intended to stand for "x belongs to y precisely z times', p. 325). This is the approach taken in [Bli 89], the most recent and sustained work on this line. Unlike many other formalizations of non-Cantorian set theory, BLIZARD [Bli 89] formulates a formal theory of msets as a conservative extension of standard set theory". BUNDER [Bun 87], using BCK-linear logic (weakened first-order logic), develops an elementary theory of msets (infact, finite msets). [Bli 91] provides an exhaustive survey of the literature dealing with the development of the theory of msets.

The term multiset as KNUTH [Knu 81, p. 36] notes, was first suggested by N. G. DE BRUIJN in a private communication to him. Owing to its aptness, it has replaced a variety of terms viz. list, heap, bunch , bag, sample , weighted set , occurrence set and fireset (finitely repeated element set) used in different contexts but conveying synonimity with mset.

As mentioned earlier, elements are allowed to repeat in an mset, finitely in most of the known application areas, albeit in a theoretical development infinite multiplicities of elements are also dealt with (see [Bli 93], [Mon 87], [Hic 80], [Lak 76], [Rad 75] and [Eil 74], in particular).

The number of copies [Bri 87, p. 5] prefers to call it 'multiples') of an element appearing in an mset is called its multiplicity. Moreover, multiple occurrences of an element in an mset are treated without preference (perhaps to retain the force of classical concept of identity). We mention [Par 81] for an earliest extensive treatment of indistinguishability of repeated elements without any preference, and [Mon 87] and [Wil 03] for an alternative treatment.

The number of distinct elements in an mset M (which need not be finite) and their multiplicities jointly determine its cardinality, denoted by $C(M)$. In other words the cardinality of an mset is the sum of multiplicities of all its elements. An mset M is called finite if the number of distinct elements in M and their multiplicities are both finite, it is infinite otherwise. Thus, an mset M is infinite if either the number of elements in M is infinite or the multiplicity of one or more of its elements is infinite i.e., $C(M) \geq \aleph_0$. The root or support or carrier of an mset M, denoted by M^* is the set containing the distinct elements of M. The elements of the root set of an mset are called the generators of that mset.

A considerable amount of efforts have also gone into the study of msets with negative multiplicities (see [Bli 90], [Hai 86], [Rei 86], [Rad 75], [Eil 74], [Whi 33], [Wil 03], in particular).

Representations of mset

1. Multiplicative form

Following Meyer and McRobbie [1982], the use of square brackets to represent an mset has become almost standard. Thus, an mset containing one occurrence of a, two occurrences of b, and three occurrences of c is notationally written as $[[a, b, b, c, c, c,]]$ or $[a, b, b, c, c, c,]$ or $[a, b, c,]_{1,2,3}$ or $[a^1, b^2, c^3]$ or $[a1, b2, c3]$, depending on one's taste and convenience.

2. Linear form

Wildberger [Wil 03] puts forward a linear notation for multisets which seems quite innovative, especially when negative multiplicities (integral as well as rational) are to be dealt with, for example, the mset $M = [a, b, c]_{1,2,3}$ can be written as $M = [a] + 2[b] + 3[c]$. Similarly, a rational mset can be represented. For example,

$$N = \frac{2}{3}[5] - \frac{1}{2}[1\_8].$$

In order to accommodate negative multiplicities round brackets are used: (a) in an mset stands for negative of a; for example,

$$[2, 4, (5), (5), 4] = [2] + 2[4] - 2[5].$$

In the same place, the distinction between the terms 'element' and 'object' occurring in an mset is made explicit as follows:

Each individual occurrence of an object x in an mset A is called an element of A. Thus in the linear notation of M above, b, for example, is an object appearing twice, and every occurrence of b is an element of M. It follows that the distinct elements of an mset are the objects. An object is an element if its multiplicity is unity.

Further, the following notations used in [Wil 03, pp. 5–6] to represent data structures of set, ordered set, multiset and list, are quite instructive:

A collection containing x, y, z, ... is denoted by $\{xyz\ldots\}$ or $\{x\_y\_z\ldots\}$ if it is a set; $\{x, y, z, \ldots\}$ if it is an ordered set; $[xyz\ldots] = [x\_y\_z\_\ldots]$ if it is a multiset; and $[x, y, z, \ldots]$ if it is a list.

Note that a list is an ordered sequence of elements with repetitions allowed, whereas an mset is a sequence with its ordering stripped off.

3. Multiset as a sequence

A multiset can also be represented as a sequence in which the multiplicity of an element equals the number of times the element occurs in the sequence, which is exactly Dedekind's 'frequency-number'. The idea is to construct an mset as a sequence (a function with domain , the set of natural numbers) and ignore the ordering of its elements, which can be done by taking all permutations of the domain of the sequence.

4. Multiset as a family of sets

A multiset can also be represented as a family of sets, which is altogether a generalization of the idea of a sequence described above. Thus, the family of sets $F = \{F_i\}$, $i \in I$, where $F_i = F_j$, if $i = j$, which identifies a repeated element, represents an mset. Clearly, such a family F is a function: $I \to \{F_i | i \in I\}$, which in turn, is a sequence if $I = \aleph_0$.

5. Multiset as a numeric-valued function

Representation of an mset as a numeric-valued or cardinal-valued function abounds, especially in the application areas. Formally, an mset is just a mapping from some ground or generic or universal set into some set of numbers. For example, an mset $\alpha = [x, y, z]_{1,2,3}$ is a mapping from a ground set S to $\aleph$, the set of natural numbers with zero, defined by

$$\alpha(t) = \begin{cases} 1, & \text{if} \quad t = x \\ 2, & \text{if} \quad t = y \\ 3, & \text{if} \quad t = z \\ 0, & \text{for all the remaining} \quad t \in S. \end{cases}$$

In general terms, for a given ground set S and a numeric set T, we call a mapping $\alpha : S \to T$,

$$\begin{cases} \text{a set, if } T = \{0, 1\}; \\ \text{a multiset, if } T = \aleph, \text{ the set of natural numbers;} \\ \text{a signed multiset (or, hybrid/shadow set) if } T = \mathbb{Z}, \text{ the set of integers;} \\ \text{a fuzzy (or hazy) set if } T = [0, 1] \subseteq \Re, \text{ a two–valued Boolean algebra.} \end{cases}$$

6. Multiset as a generalized characteristic function

Similar to the representation of a set by its characteristic function (function whose range is $\{0, 1\}$), a multiset or hybrid set is determined by its generalized characteristic function (whose range is the set of integers, positive, negative or zero), see [Whi 33] for details.

Operations under Mset. [Knu 81] can be considered as the earliest reference describing intuitively properties of msets in a sufficient detail. During the recent years, a good number of papers ([Lak 76], [Hai 86], [Hic 80], [Bli 89], [Wil 03], and others) have appeared. We endeavour to present an overview of various approaches in this regard. We will adhere to function–approach and use $\text{Dom}(f), \text{Ran}(f)$ to denote the domain and range respectively of a given function f.

Definition 1. **Multiset**. Let $D = \{x_1, x_2, \ldots, x_j, \ldots\}$ be a set. An mset A over D is a cardinal-valued function i.e., $A : D \to \aleph = \{0, 1, 2, ...\}$ such that for $x \in \text{Dom}(A)$ implies $A(x)$ is a cardinal and $A(x) = m_A(x) > 0$, where $m_A(x)$ denotes the number of times an object x occurs in A, i.e., a counting function of A. The set D is called the ground or generic set of the class of all msets containing objects from D.

An mset A can also be represented by the set of pairs as follows:

$$A = \{\langle m_A(x_1), x_1\rangle, \ldots, \langle m_A(x_j), x_j\rangle, \ldots\}$$

or,

$$A = \{m_A(x_1) \cdot x_1, \ldots, m_A(x_j) \cdot x_j, \ldots\}$$

Relatedly, an mset is called 'regular' or 'constant' if all its elements occur with the same multiplicity. Also an mset is called 'simple' if all its elements are the same, for example, $[x]_3$ is a simple mset containing x as its only object.

Definition 2. **Dressed epsilon symbol,** $\in_+$. The symbol $\in_+$ was first introduced by Singh & Singh [SS 07]. For any object x occurring as an element of an mset A i.e., $m_A(x) > 0$, we write $x \in_+ A$, where $\in_+$ (dressed epsilon is a binary predicate intended to be 'belongs to at least once', as $\in$ is 'belongs to only once' in the case of sets). Thus, $m_A(x) = 0$ implies $x \in A$, and $x \in^k_+ A$ implies 'x belongs to A at least k times', however $x \in^k A$ means 'x belongs k times to A'. The mset for any ground set D is called empty, denoted by $\varnothing$ or [ ], if $m_\varnothing(x) = 0$ for all $x \in D$.

Further, in order to make our presentation concise, we shall follow some terminologies introduced in [Hic 80, pp. 212–213]: $A(x)$ denotes the number of copies of x, including x itself, belonging to Dom(A), which is exactly the Dedekind's frequency number.

Definition 3. Equal msets. Two msets A and B are equal or the same, written as $A = B$, iff for any object $x \in D$, $m_A(x) = m_B(x)$ or $A(x) = B(x)$. Equivalently, $A = B$ if every element of A is in B and conversely. Clearly, $A = B \Rightarrow A^* = B^*$, however the converse need not hold.

Definition 4. Multisubsets (or **msubsets**, for short). Let A and B be two msets, A is an msubset or a submultiset of B, written as $A \subseteq B$ or $B \supseteq A$, if $m_A(x) \leq m_B(x)$ for all $x \in D$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a *proper submset of* B. An mset is called the *parent* in relation to its msubsets.

It is easy to see that $\subseteq$ is antisymmetirc i.e., $A \subseteq B$ and $B \subseteq A \Rightarrow A = B$, and it is a partial ordering on the class of msets defined on a given generic domain. Clearly, $\varnothing$ is a submset of every mset.

Note that the terms 'element' and 'object' are being distinguished throughout, and coincide if a generic set is in consideration. We wish to emphasize that introduction of $\in_+$ greatly enhance the language of msets. For example, $A \subseteq B$ stands for

$$\forall z \forall k (z \in^k A \Rightarrow z \in^k_+ B).$$

Relatedly, a 'whole' msubset of a given mset contains all multiplicities of common elements; while a 'full' msubset contains all objects of the parent mset and accordingly, every mset contains a unique full msubset, called its root set.

Clearly, for any two msets A and B, if $A \subseteq B$ and $\text{Dom}(A) = \text{Dom}(B)$, then A is a *full msubset* of B.

Definition 5. Similar msets. Two msets A and B are said to be 'cognate' or similar if $\forall x(x \in A \Rightarrow x \in B)$, where x is an object. Thus, similar msets have equal root sets but need not be equal themselves.

Definition 6. Ordered pair of two mset terms. Ordered pair of two mset terms u and v, denoted by $[u, v]$, can be defined as follows:

$$[u, v] = \{u, v\} \,\text{if}\, u \neq v, \text{ and } [u, v] = \{[u]_2\} \,\text{if}\, u = v.$$

Here, $[u]$ is written as $\{u\}$, and $\langle u, v\rangle$ is actually the ordered pair set, where set (u) stands for $u = \varnothing \vee \forall x \forall n(x \in^n u \Rightarrow n = 1)$, though x itself may be an mset term(see [Bli 89, pp. 42–44], for details).

Definition 7. Power multiset. In Cantorian spirit, the power multiset of a given mset A, denoted by $\tilde{\wp}(A)$ to distinguish it from the symbol $\wp(A)$ used for power set of A, is the multiset of all submultisets of A. For example, let $A = [x, y]_{2,1} = [x, x, y]$. Then, $\tilde{\wp}(A) = [\varnothing, \{x\}, \{x\}, [x]_2, \{y\}, \{x, y\}, \{x, y\}, [x, y]_{2,1}]$. In this sense, $C(\tilde{\wp}(A)) = 2^{C(A)}$, for any mset A.

However, as has been voiced by many researchers in the area of msets and their applications (see [Hic 80, p. 213] and [Bli 89, p. 45], in particular), there is no 'good' reason for admitting repeated elements into a power multiset. Hence, a power multiset needs to be called a power set only and denoted by $\wp(A)$. Accordingly, for $A = [x, y]_{2,1}$, and hence $C(\wp(A)) < 2^{C(A)}$ which implies that Cantor's power set theorem: $C(A) < C(\wp(A))$ fails. However, for finite msets, Cantor's theorem holds for power mset (see [Bli 89, p. 45], for related inherent difficulties if the mset in consideration is infinite).

Definition 8. Union ($\cup$), intersection ($\cap$) and addition or sum or merge ($+$) or $\uplus$. Let A and B be two msets over a given domain set D.

1. $A \cup B$ is the mset defined by

$$m_{A\cup B}(x) = m_A(x) \cup m_B(x) = \text{maximum}(m_A(x), m_B(x)),$$

being the union of two numbers. That is, an object z occurring a times in A and b times in B, occurs maximum(a, b) times in $A \cup B$, if such a maximum exists; otherwise the minimum of (a, b) is taken which always exists.

It follows that for any given mset x there exists an mset y which contains elements of elements of x, where the multiplicity of an element z in y is the maximum multiplicity of z as an element of elements of x along with the above stipulation on the existence of such a maximum. We denote this fact by $y = \cup x$.

Clearly, $\text{Dom}(\cup x) = \cup\{\text{Dom}(A), A \in x\}$ and that the multipliticity of z in y is the maximum of its multiplicities as an element of elements of x, if it exists, otherwise, the minimum is taken.

For example, if $A = [2\,3\,4\,4]$, $B = [1\,4\,3\,3]$ then $A \cup B = [1\,2\,3\,3\,4\,4]$.

Also, it follows that for a finite mset x, the maximum multiplicity of elements of elements of x always exists. However, for certain infinite sets like $x = \{y, [y]_2, [y]_3, \ldots\}$, the maximum multiplicity of elements of elements of x does not exist, and hence $x = \cup\{y\}$. It is obvious by definition of the union that multiplicity of any $y \in x \neq \varnothing$ is irrelevant to $\cup x$, and hence $\cup x = \cup x^*$ (see [Bli 89, pp. 48–49], for details).

2. $A \cap B$ is the mset defined by

$$m_{A\cap B} = m_A \cap m_B = \text{minimum}(m_A(x), m_B(x),$$

being the intersection of two numbers.

That is, an object x occurring a times in A and b times in B, occurs minimum(a, b) times in $A \cap B$, which always exists.

In general, for a given mset x, $\text{Dom}(\cap x) = \cap\{\text{Dom}(A) : A \in x\}$ and $z \cap x$ implies that the multiplicity of z is the minimum of its multiplicities as element of elements of x.

For example, if $A = [3\,3\,3\,4\,4]$, $B = [1\,4\,3\,3]$, then $A \cap B = [3\,3\,4]$. Note that for any mset x, we have $\cap x \subseteq \cup x$.

3. $A + B$ or $A \uplus B$ is the mset defined by $m_{A+B}(x) = m_A(x) + m_B(x)$, for any $x \in D$, direct sum of two numbers.

That is, an object x occurring a times in A and b times in B, occurs $a + b$ times in $A \uplus B$.

For example if $A = [1\,1\,2\,2\,4\,4\,4]$, $B = [1\,2\,3\,3]$ then $A \uplus B = [1\,1\,1\,2\,2\,2\,3\,3\,4\,4\,4]$. Clearly, $C(A \uplus B) = C(A \cup B) + C(A \cap B)$. Note that if x be an infinite mset, then the multiplicity of some mset $z \in \uplus x$ may not be finite. In that case, the multiplicity of z in x is used, For example, if $x = \{\{z\}, [z]_2, [z]_3 \ldots\}$ then $\uplus x = \cup x = \{z\}$ (see [Bli 89, p. 51], for details).

4. Some Properties holding for mset operations (see [Knu 81, p. 636]; [Bli 89, p. 53], in particular).

(i) Commutativity: $A \uplus B = B \uplus A$, $A \cup B = B \cup A$, $A \cap B = B \cap A$.

(ii) Associativity:

$$A \uplus (B \uplus C) = (A \uplus B) \uplus C\,,$$
$$A \cup (B \cup C) = (A \cup B) \cup C\,,$$
$$A \cap (B \cap C) = (A \cap B) \cap C.$$

(iii) Idempotency:

$$A \cup A = A\,, A \cap A = A, \text{ but } A \uplus A \neq A\,.$$

In fact, as it has been suggested recently (see [Wil 03, p. 9], in particular), in order to obtain a linear combination of msets, kA may be interpreted to denote the sum of k number of A's, where k is a natural number.

(iv) Identity laws:

$$A \cup \varnothing = A,\ A \cap \varnothing = \varnothing,\ A \uplus \varnothing = A\,.$$

(v) Distributivity:

$$A \uplus (B \cup C) = (A \uplus B) \cup (A \uplus C),$$
$$A \uplus B \cap C) = (A \uplus B) \cap (A \uplus C),$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

The proof of all these identities follows from the interpretation of $\cup$, $\cap$ and $\uplus$ of two natural numbers as maximum, minimum and (direct) sum respectively. It is easy to see that $\uplus$ is stronger than both $\cup$ and $\cap$ in the sense that neither nor distributes over $\uplus$, whereas $\uplus$ distributes over both $\cup$ and $\cap$. Also,

$$\cap x \subseteq \cup x \subseteq \uplus x\,.$$

It is promising to observe that multiset operations form a "realm" [Wil 03, p. 9].

Definition 9. Difference and complementation. Let A and B be two msets over D, and $B \subseteq A$, then $m_{A-B}(x) = m_A(x) - m_{A\cap B}(x)$, for all $x \in D$.

It is sometimes called the *arithmetic difference* of B from A. Note that even if $B \subsetneq A$, this definition holds good.

It can be seen quickly that some of the consequences of the aforesaid definition are disturbing [Hic 80, p. 214].

For example, if $A = [a, b]_{4,5}$ and $B = [a, b]_{2,3}$ then $A - B = [ab]_{2,2} \subseteq B$, contradicting the classical law: $(A - B) \cap B = \varnothing$.

In order to define the complement, we follow PETROVSKY [Pet 97, pp. 3–4]:

Let $\Im = \{A_1, A_2, \ldots\}$ be a family of multisets composed of the elements of the generic set D. Then, the maximum multiset z is defined by

$$m_z(x) = \max_{A\in\Im} m_A(x)\,,$$

for all $x \in D$ and all $A \in \Im$. Now, the complement of an mset A, denoted by $\overline{A}$, is defined as follows:

$$\overline{A} = Z - A = \{m_{\overline{A}}(x) \cdot x \mid m_{\overline{A}}(x) = m_Z(x) - m_A(x)\,, \forall\, x \in D\}\,.$$

It is understood that some new operations like arithmetic multiplication, raising to the arithmetic power, direct product, raising to the direct power, defined by PETROVSKY, can be gainfully exploited for further research.

Definition 10. Functions between msets. The underlying assumption in defining a function between msets has been invariably not to allow mapping of identical elements to non-identical elements and hence, it amounts to defining the function between their root sets, which is just the classical definition of a function.

The function $f : A \to B$ is an injection iff

(i) $f : A^* \to B^*$ is an injection, and
(ii) $\forall z(z \in A^* \Rightarrow m_A(z) \leq m_B(f(z)))$.

The function $f : A \to B$ is a surjection iff

(i) $f : A^* \to B^*$ is a surjection, and
(ii) $\forall z(z \in A^* \Rightarrow m_A(z) \geq m_B(f(z)))$.

The function f: A B is a bijection iff

(i) $f : A^* \to B^*$ is a bijection and
(ii) $\forall z(z \in A^* \Rightarrow m_A(z) = m_B(f(z)))$.

For example, $f : [x]_3 \to [x]_{10}$ is an injection, $f : [x]_5 \to [y]_4$ is a surjection, and $f : [x, y]_{6,2} \to [x, y]_{3,4}$ is neither an injection nor a surjection. For various other details, see [Bli 89] and [Hic 80].

Note that some of the consequences of the aforesaid definitions are conflicting with some fundamental theorems of the classical set theory.

1. Having defined functions between msets as above, it can be proved that Cantor's theorem does not hold, there is no injection from $A \to \wp(A)$, see [Hic 80, p. 215].

2. Msets of equal cardinality need not have a bijection between them. For example, $[a, b]_{1,2}$ and $[a, b, c]$ both contain three elements, but there can be no bijection between them because the objects and their multiplicities are different in the domain and the range of any such function. In the other words, there is no bijection between their root sets viz: $f : \{a, b\} \to \{a, b, c\}$ can not be a bijection.

3. Schröder-Bernstein theorem fails (see [Hic 80, p. 215] and [Bli 89, p. 47]). Let $A = [x_1, x_2, \ldots]_{2,4,6,\ldots}$ and $B = [y_0, y_1, y_2, \ldots]_{1,3,5,\ldots}$ The function $f : A* \to B*$ defined as $f(x_n) = y_n$ makes $f : A \to B$ an injection so that $A \leq B$. The function $g : B* \to A*$ defined by $g(y_n) = x_{n+1}$ makes $g : B \to A$ an injection so that $B \leq A$. But there cannot be a bijection $h : A \to B$ since all multiplicities in A are even and that in B are odd. Note that $\leq$ is the standard dominance relation.

Definition 11. Multiset ordering. It seems really surprising that the seminal work of KNUTH [Knu 73, pp. 213–214, 241–242], related to multiset orderings and their applications, has escaped the attention of most of us until quite recently. According to KNUTH:

> "Multiset μ_1 dominates μ_2 if both μ_1 and μ_2 contain the same number of elements and the K^{th} largest element of μ_1 is greater than or equal to the K^{th} largest element of μ_2 for all K" (p. 214).

> "If a and b are multisets of m numbers each, we say that $a \ll b$ iff $a \hat{\wedge} b = a$ (equivalently, $a \check{\vee} b = b$, the largest element of a is less than or equal to the smallest of b). Thus $a \hat{\wedge} b \ll a \check{\vee} b$"(p. 241).
>
> "An n^{th} level 'cascade distribution' is a multiset ..." (p. 299).

In fact, on a serious note, it is out and out a pioneering work on multiset ordering and its applications. However, [DM 79] is the earliest reference known to introducing multiset ordering and using it for proving termination of programs and term rewriting systems. In fact, it has served as a basis for host of orderings introduced in this context. We endeavour to outline the Dershowitz-Manna multiset ordering as follows:

Let S be a set equipped with a partial ordering $<$ (irreflexive and transitive relation or, equivalently, a transitive but not an equivalence relation). Let $M(S)$ be the set of all the finite msets M on S, and let $\ll$ be the associated (induced by $<$) mset ordering on $M(S)$. It is easy to see that each M is an mset with a finite carrier viz; $\{x \in S : M(x) \neq 0\}$.

The Dershowtiz-Manna ordering. $M \ll N$ if there exist two msets X and Y in $M(S)$ satisfying:

(i) $\{\ \} \neq X \subseteq N$,
(ii) $M = (N - X) + Y$,
(iii) $(\forall y \in Y)(\exists x \in X)[y < x]$

In other words, $M \ll N$ if M is obtained from N by removing none or at least one element (those in X) from N, and replacing each such element x by zero or any finite number of elements (those in Y), each of which is strictly less than (in the ordering $<$) one of the elements x that have been removed. Informally, we say that M is smaller than N in this case. Similarly, $\gg$ on $M(S)$ with $(S, >)$ can be defined.

For example, let $S = (\{0, 1, 2, \ldots\} = \aleph)$, then under the corresponding multiset ordering $\gg$ over $\aleph$, the mset [3 3 4 0] is greater than each of the following msets: [3 4], [3 2 2 1 1 1 4 0] and [3 3 3 3 2 2]. The empty set { } is smaller than any multiset.

It is also easy to observe that:

$$[(\forall y)(y \in N) \Rightarrow (\exists x)(x \in M \wedge x > y)] \Rightarrow M \gg N.$$

For various ramifications of the Dershowitz-Manna ordering, see [JL 82] and [Mar 89], in particular.

Acknowledgement: The authors are thankful to the anonymous referees for their constructive comments and valuable suggestions, and grateful to the editor for his patience and coordination in implementing corrections in the galley proof of the paper.

References

[Ang 65] Angelelli, I., *Leibniz's misunderstanding of Nizolius notion of 'Multitudo'*, Notre Dame Journal of Formal Logic **7**(4) (1965), 319–322.

[Bli 89] Blizard. W., *Multiset theory.* Notre Dame Journal of Formal Logic **30** (1989), 36–66.

[Bli 90] Blizard. W., *Negative membership.* Notre Dame Journal of formal logic **31** (1990), 346–368.

[Bli 91] Blizard. W., *The Development of Multiset Theory.* Modern Logic **1** (1991), 319–352.

[Bli 93] Blizard. W., *Dedekind multisets and function shells.* Theoretical Computer Science **110** (1993), 79–98.

[Bri 78] Brink C., *On Peirce's notation for the Logic of relatives.* Transactions of the Charles S. Peirce Society. (1978), 285–304.

[Bri 87] Brink C., *Some backgrounds on Multisets*, TR - ARP - 2/87, ANU, Canberra, Australia, 1987.

[Bun 87] Bunder M. V., *A Foundation for Multiset Theory (using BCK - linear logic)*, Department of Marthematics, University of Wollongong. TR, 1987.

[Bun 83] Bundy A., *The Computer Modelling for Mathematical Reasoning.* Academic press, New York, 1983.

[Can 55] Cantor G., *Contributions to the Ffounding of the tTheory of Transfinite Numbers.* Dover, New York, 1955 (English translation with introduction and notes by P. Jourdan).

[Cor 80] Corcoran J., *Categoricity.* History and Philosophy of Logic, **1** (1980), 187–207.

[DM 79] Dershowitz N. & Manna Z., *Proving termination with multiset orderings.* Comm. ACM **22** (1979), 465–476.

[Eil 74] Eilenberg S., *Automata, Languages and Machines*, Vol. A, Academic Press, New York, 1974.

[FBL 73] Fraenkel, A., Bar-Hillel Y. & Levy A., *Foundations of Set theory.* Second revised edition, North Holland, 1973.

[Grz 87] Gryzymala-Busse J. W., *Learning from Examples based on rough multisets*, in Z. W. Ras & M. Zeamankova (editors), *Methodologies for Intelligent Systems*, North Holland, 325–332, 1987.

[Hai 86] Hailperin T., *Boole's logic and probability*, Second edition, North Holland, 1986.

[Hal 84] Hallet M., *Cantorian Set Theory and Limitation of Size*, Clarendon Press, Oxford, 1984.

[Hic 80] Hickman J.L., *A Note on the concept of Multiset*, Bulletin of the Australian Mathematical society **22** (1980) 211–217.

[Hig 52] Higman G., *Ordering by Divisibility in an Abstract Algebra.* Proceedings, London Mathematical Society **2** (3) (1952), 326–336.

[Ifr 85] Ifrah G., *From one to zZero: A Universal History of Numbers.* Viking Penguin, New York, 1985.

[JL 82] J.-P. Jouannaud & P. Lescanne. *On multiset orderings.* Information Processing letters **15** (1982), 57–63.

[Knu 73] Knuth D. E, *The Art of Computer Programming*, Vol. 3, (Sorting and Searching). Addison-Wesley, Reading, Mass., 1973.

[Knu 81] Knuth D. E., *The Art of Computer Programming*, Vol. 2 (Seminumerical Algorithms), Second Edition. Addison-Wesley, Reading, Mass., 1981.

[Lak 76] Lake J., *Sets, Fuzzy Sets, Multisets and Functions.* Journal of the London Math. Society **12** (2) (1976), 323–326.

[Mar 89] Martin U., *A geometrical approach to Multiset orderings.* Theoretical Computer Science **67** (1989), 37–54.

[MM 82] MEYER R. K. & MCROBBIE M. A., *Multisets and Relevant Implication I and II*. Australasian Journal of Philosophy **60** (1982), 107–139 and 265–281.

[Mon 87] *Monro G. P.*, *The Concept of Multiset*. Zeitschrift fur Mathematische Logik und Grundlagen der Mathematik **3** (1987), 171–178.

[Par 81] PARKER–RHODES A. F. *The theory of indistinguishables*. D. Reidel, Dordrecht, 1981.

[Pet 97] PETROVSKY A. B., *Structuring techniques in multiset spaces*. Springer, 174–84, 1997.

[Rad 75] RADO R., *The cardinal module and some theorems on families of sets*. Annali di Matematica pura ed applicata **102** (4) (1975), 135–154.

[Rei 86] REISIG W., *Petri Nets: An Introduction*, Springer-Verlag, Berlin, 1986.

[Sin 94] SINGH D., *A note on the development of multiset theory*. Modern Logic **4** (1994),405–406.

[SS 07] SINGH D. & SINGH J. N., *A note on the definition of multisubset*, Association for Symbolic Logic, Jan 7–8, 2007.

[Wey 49] WEYL H., *Philosophy of Mathematics and Natural Science*. Princeton University Press, Princeton, NJ, 1949.

[Whi 33] WHITNEY H., *Characteristic functions and the algebra of Llogic*. Annals of Mathematics **34** (1933), 405–414.

[Wil 03] WILDBERGER N. J., *A new look at multiset*. School of Mathematics, UNSW Sydney 2052, Australia, 2003.

[Yag 86] YAGER R. R., *On the theory of bags*. International Journal of General Systems **13** (1986), 23–37.

[Yag 87] YAGER R. R., *Cardinality of fuzzy sets via bags*. Mathematical Modelling **9** (1987), 441–446.

(Recibido en junio de 2007. Aceptado para publicación en noviembre de 2007)

D. SINGH, DEPARTMENT OF MATHEMATICS
AHMADU BELLO UNIVERSITY, ZARIA, NIGERIA
e-mail: mathdss@yahoo.com

A. M. IBRAHIM, DEPARTMENT OF MATHEMATICS
AHMADU BELLO UNIVERSITY, ZARIA, NIGERIA
e-mail: adekubash@yahoo.com

T. YOHANNA, DEPARTMENT OF MATHEMATICS,
KADUNA STATE UNIVERSITY, KADUNA, NIGERIA
e-mail: yot2002ng@yahoo.com

J. N. SINGH, DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
BARRY UNIVERSITY, MIAMI SHORES, FL 33161, USA
e-mail: jsingh@mail.barry.edu