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Extracting the roots of septics by polynomial decomposition

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Abstract. A method is described to solve septic equations in radicals. The salient feature of such a solvable septic is that the sum of its four roots is equal to the sum of its remaining three roots. The conditions to be satisfied by the coefficients of the septic are given. A numerical example is solved using the proposed method.

Key words and phrases. Septic equation, octic equation, seventh-degree polynomial equation, polynomial decomposition.

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Resumen. Se desribe un método para resolver ecuaciones de grado siete mediante radicales. La característica sobresaliente de estas ecuaciones de grado siete es que la suma de de cuatro de sus raíces es igual a la suma de las tres restantes. Se dan las condiciones que deben satisfacer los coeficientes de la ecuación y se resuelve con este método un ejemplo numérico.

1. Introduction

Several mathematicians have tried to solve general polynomial equations of degree higher than four with the techniques similar to that applied to solve cubics and quartics. However they didn't succeed. In 1770 Lagrange showed that these equations couldn't be solved with such methods. Later Paolo Ruffini (1799), Abel (1826), and Galois (1832), proved more rigorously that it is impossible to solve the general polynomial equations of degree five and above in radicals. With certain conditions imposed on the coefficients (or

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equivalently on the roots), these equations become solvable in radicals, and are aptly named solvable equations ([1], [2]).

In this paper we present a method to solve a septic equation which has the property that the sum of its four roots is equal to the sum of its remaining three roots. In the method proposed here the septic equation is first converted to an octic equation by adding a root; the octic is then decomposed into two quartic polynomials in a novel fashion. The quartic polynomial factors are equated to zero and solved to obtain the seven roots of the given septic along with the added root. In the next section we describe the decomposition method, and in the further sections we discuss the behavior of roots and the conditions for the coefficients. In the last section a numerical example is solved using the proposed method.

2. **Decomposition method**

Consider the following septic equation, for which a solution in radicals is sought:

$$x^7 + a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0\,. \tag{1}$$

Here a_0, a_1, a_2, a_3, a_4, a_5, and a_6 are rational coefficients. As a first step, the equation above is multiplied by x to convert it into an octic equation as shown below:

$$x^8 + a_6x^7 + a_5x^6 + a_4x^5 + a_3x^4 + a_2x^3 + a_1x^2 + a_0x = 0\,. \tag{2}$$

This process [of multiplying (1) by x] implies that a root at the origin is added to the septic equation (1). Consider another octic equation as shown below:

$$[(x^4 + b_3x^3 + b_2x^2 + b_1x + b_0)^2 - (c_3x^3 + c_2x^2 + c_1x + c_0)^2] = 0\,, \tag{3}$$

where b_0, b_1, b_2, b_3 and c_0, c_1, c_2, c_3 are unknown coefficients in the quartic and cubic polynomials respectively, in equation (3). Notice that the octic equation (3) can be decomposed into two factors as shown below:

$$[x^4 + (b_3 - c_3)x^3 + (b_2 - c_2)x^2 + (b_1 - c_1)x + b_0 - c_0]\times$$
$$[x^4 + (b_3 + c_3)x^3 + (b_2 + c_2)x^2 + (b_1 + c_1)x + b_0 + c_0] = 0\,. \tag{4}$$

Therefore, if the octic equation (2) can be represented in the form of (3), then it can be factored into two quartic factors as shown in (4), leading to its solution. These quartic factors are equated to zero to obtain the following quartic equations:

$$\begin{aligned} x^4 + (b_3 - c_3)x^3 + (b_2 - c_2)x^2 + (b_1 - c_1)x + b_0 - c_0 = 0 \\ x^4 + (b_3 + c_3)x^3 + (b_2 + c_2)x^2 + (b_1 + c_1)x + b_0 + c_0 = 0\,. \end{aligned} \tag{5}$$

The seven roots of the given septic equation (1) along with the added root are then obtained by solving the above quartic equations. In order to represent the octic equation (2) in the form of (3), we expand (3) and equate the corresponding coefficients. The expansion of (3) in descending powers of x is shown below:

$$\begin{aligned} &x^8 + 2b_3x^7 + (b_3^2 + 2b_2 - c_3^2)x^6 + 2(b_1 + b_2b_3 - c_2c_3)x^5 \\ &\quad +[b_2^2 + 2b_0 + 2b_1b_3 - (c_2^2 + 2c_1c_3)]x^4 + 2[b_0b_3 + b_1b_2 - (c_0c_3 + c_1c_2)]x^3 \\ &\qquad +[b_1^2 + 2b_0b_2 - (c_1^2 + 2c_0c_2)]x^2 + 2(b_0b_1 - c_0c_1)x + b_0^2 - c_0^2 = 0\,. \end{aligned} \tag{6}$$

Equating the coefficients of (6) with the coefficients of octic equation, (2), we obtain the following eight equations:

$$2b_3 = a_6 \tag{7}$$

$$b_3^2 + 2b_2 - c_3^2 = a_5 \tag{8}$$

$$2(b_1 + b_2b_3 - c_2c_3) = a_4 \tag{9}$$

$$b_2^2 + 2b_0 + 2b_1b_3 - (c_2^2 + 2c_1c_3) = a_3 \tag{10}$$

$$2[b_0b_3 + b_1b_2 - (c_0c_3 + c_1c_2)] = a_2 \tag{11}$$

$$b_1^2 + 2b_0b_2 - (c_1^2 + 2c_0c_2) = a_1 \tag{12}$$

$$2(b_0b_1 - c_0c_1) = a_0 \tag{13}$$

$$b_0^2 - c_0^2 = 0\,. \tag{14}$$

Even though there are eight equations [(7) to (14)] and eight unknowns (b_0, b_1, b_2, b_3, c_0, c_1, c_2, and c_3), to be determined, we need one more equation called the *supplementary equation* to find all the unknowns and thereby make the septic equation (1) solvable in radicals. The supplementary equation introduced will decide the type of solvable septic. Let the supplementary equation introduced be as follows:

$$c_3 = 0\,. \tag{15}$$

Now we start the process of determining the unknowns using the equations, (7) to (15) in a sequential manner. First consider equation (7) from which b_3 is evaluated as:

$$b_3 = a_6/2 \tag{16}$$

Using the values of b_3 and c_3 [from (16) and (15) respectively] in equation (8) b_2 is found to be:

$$b_2 = a_7 \tag{17}$$

where a_7 is given by:

$$a_7 = (a_5/2) - (a_6^2/8).$$

Using the values of b_2, b_3, and c_3 [from (17), (16), and (15)] in equation (9), b_1 is evaluated as:

$$b_1 = a_8\,, \tag{18}$$

where $a_8 = (a_4 - a_6a_7)/2$. In the same manner, we use the values of b_1, b_2, b_3, and c_3 [from (18), (17), (16), and (15)] in equation (10) to obtain an expression for b_0:

$$b_0 = a_9 + (c_2^2/2)\,, \tag{19}$$

where $a_9 = (a_3 - a_6 a_8 - a_7^2)/2$. Next, the values of b_1, b_2, b_3, and c_3 given by equations (18), (17), (16), and (15), are substituted into equation (11) to yield

$$2c_1 c_2 = 2a_7 a_8 - a_2 + a_6 b_0\,. \tag{20}$$

Similarly, when the values of b_1 and b_2 given in (18) and (17) are substituted into (12), we obtain

$$c_1^2 = a_8^2 - a_1 + 2a_7 b_0 - 2c_0 c_2\,. \tag{21}$$

The value of b_1 [from (18)] is used in (13) to get the following expression:

$$2a_8 b_0 - 2c_0 c_1 = a_0\,. \tag{22}$$

From (14) we obtain two expressions for c_0: $c_0 = \pm b_0$, and we choose one of them as indicated below.

$$c_0 = b_0\,. \tag{23}$$

Using the expressions (19) and (23), we eliminate b_0 and c_0 from the expressions (20), (21), and (22) to obtain the following three expressions in the unknowns c_1 and c_2:

$$2c_1 c_2 = a_{10} + (a_6 c_2^2/2)\,, \tag{24}$$

where a_{10} is given by

$$\begin{aligned} a_{10} &= a_6 a_9 + 2a_7 a_8 - a_2, \\ c_1^2 &= a_{11} - 2a_9 c_2 + a_7 c_2^2 - c_2^3\,, \end{aligned} \tag{25}$$

where a_{11} is given by

$$a_{11} = a_8^2 + 2a_7 a_9 - a_1\,,$$

and

$$a_8 c_2^2 - (c_2^2 + 2a_9) c_1 + a_{12} = 0\,, \tag{26}$$

where a_{12} is expressed as

$$a_{12} = 2a_8 a_9 - a_0\,.$$

Using (24) we eliminate c_1 from the expressions (25) and (26) to obtain the following two expressions in c_2:

$$c_2^5 + [(a_6^2 - 16a_7)/16]c_2^4 + 2a_9 c_2^3 + [(a_6 a_{10} - 4a_{11})/4]c_2^2 + (a_{10}^2/4) = 0\,, \tag{27}$$

$$a_6 c_2^4 - 4a_8 c_2^3 + 2(a_6 a_9 + a_{10})c_2^2 - 4a_{12} c_2 + 4a_9 a_{10} = 0\,. \tag{28}$$

The quartic equation (28) yields four values of c_2; and the one satisfying the quintic equation (27) is the desired value of c_2, which is used to determine the remaining unknowns, b_0, c_0, and c_1. Thus expression (19) is used to determine b_0 (and hence c_0), and the expression (22) is used to determine c_1. With the determination of all unknowns, the quartic equations given in (5) are simplified as:

$$x[x^3 + (a_6/2)x^2 + (a_7 - c_2)x + a_8 - c_1] = 0\,, \tag{29}$$

$$x^4 + (a_6/2)x^3 + (a_7 + c_2)x^2 + (a_8 + c_1)x + 2b_0 = 0\,. \tag{30}$$

The factor, x, in the quartic equation (29) indicates the added root (at the origin) to the given septic equation (1). Equating the cubic factor in (29) to zero results in the cubic equation

$$x^3 + (a_6/2)x^2 + (a_7 - c_2)x + a_8 - c_1 = 0\,. \tag{31}$$

Solving the quartic equation (30) and the cubic equation (31) using the methods given in literature [3, 4], we obtain all the seven roots of the given septic equation (1). Thus we have shown that the septic equation can be solved in radicals (under certain conditions) using the proposed method. The behavior of the roots and the conditions on the coefficients of such solvable septics are discussed in the coming sections.

3. Behavior of roots

Let x_1, x_2, x_3, and x_4 be the roots of the quartic equation (30); x_5, x_6, and x_7 be the roots of the cubic equation (31). Observing the equations (30) and (31), we note that the coefficient of x^3 in the quartic equation (30) is same as the coefficient of x^2 in the cubic equation (31), which means the sum of the roots of (30) is equal to the sum of the roots of (31) as indicated below:

$$x_1 + x_2 + x_3 + x_4 = x_5 + x_6 + x_7\,. \tag{32}$$

Thus the solvable septic equation (1) has the property that the sum of its four roots is equal to the sum of its remaining three roots.

4. Conditions for coefficients

In section 2 we obtained two equations, one quintic (27) and one quartic (28) in the unknown, c_2. A close observation of these two equations reveals that the quintic (27) contains the coefficients, a_1, a_2, a_3, a_4, a_5, and a_6, while quartic (28) contains the coefficients, a_0, a_2, a_3, a_4, a_5, and a_6. The quartic equation (28) is used to extract four values of c_2 in radicals, whereas the quintic equation (27) is used to pick the desired value of c_2 from the four values. Thus the desired value of c_2 is nothing but a common root of (27) and (28). Therefore it is obvious that at least one non-zero real root of quartic (28) should satisfy the quintic equation (27) in order to solve the given septic equation by the proposed method. If there is more than one non-zero real root common to (28) and (27), then there is more than one way to arrange the roots of the septic so that (32) is satisfied.

5. Synthesis of solvable septics

We now attempt to synthesize the septic equations solvable by the given method. Since the quartic (28) does not contain a_1, it is chosen as dependent coefficient, so that it can be determined from the remaining coefficients, a_0, a_2, a_3, a_4, a_5, and a_6. The quartic (28) is formed with these coefficients and is solved to obtain its real roots. The real roots are then used in quintic (27) to obtain a_1 as shown below:

$$a_1 = [a_8^2 + 2a_7a_9 - (a_6a_{10})/4] - \{c_2^5 + [(a_6^2 - 16a_7)/16]c_2^4 + 2a_9c_2^3 + (a_{10}^2/4)\}/c_2^2\,. \tag{33}$$

Thus for each set of coefficients $\{a_0, a_2, a_3, a_4, a_5, a_6\}$, there can be at the most four values of a_1 [if all the roots of (28) are real], and one can form four septic equations solvable with this method.

Observe that when $c_2 = 0$, there is division by zero in (33), rendering the method to fail. Moreover, when $a_0 = a_2 = a_4 = a_6 = 0$, the quartic in (28) is identically zero, which also causes the method to fail. On the other hand, if at least one of a_2, a_4, or a_6 is non-zero, then the quartic in (28) is non-zero. In view of these results, we propose the following two theorems.

Theorem A. *Suppose*

$$p(x) = x^7 + a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$

is a polynomial with rational coefficients. Let

$$a_7 = (a_5/2) - (a_6^2/8)\,; \qquad a_8 = (a_4 - a_6a_7)/2\,;$$
$$a_9 = (a_3 - a_6a_8 - a_7^2)/2 \qquad a_{10} = a_6a_9 + 2a_7a_8 - a_2\,;$$
$$a_{11} = a_8^2 + 2a_7a_9 - a_1; \qquad a_{12} = 2a_8a_9 - a_0\,,$$

and consider the equations

$$a_6x^4 - 4a_8x^3 + 2(a_6a_9 + a_{10})x^2 - 4a_{12}x + 4a_9a_{10} = 0 \tag{A1}$$
$$x^5 + [(a_6^2 - 16a_7)/16]x^4 + 2a_9x^3 + [(a_6a_{10} - 4a_{11})/4]x^2 + (a_{10}^2/4) = 0 \tag{A2}$$

Suppose that at least one of the coefficients a_2, a_4, a_6 is non-zero, and suppose that $c \neq 0$ is a common root of (A1) and (A2), which is rational. Put $b = a_9 + (c^2/2)$, and $d = (a_{10}/2c) + (a_6c/4)$. Then

$$p(x) = [x^3 + (a_6/2)x^2 + (a_7 - c)x + a_8 - d][x^4 + (a_6/2)x^3 + (a_7 + c)x^2 + (a_8 + d)x + 2b].$$

Theorem B. *Assuming the notation of Theorem A, let $a_0, a_1, a_2, a_3, a_4, a_5, a_6$ be rational numbers. Suppose that a_2, a_4, a_6 are not zero simultaneously, and suppose that c is a non-zero real root of (A1). Put*

$$a_1 = [a_8^2 + 2a_7a_9 - (a_6a_{10})/4] - \{c^5 + [(a_6^2 - 16a_7)/16]c^4 + 2a_9c^3 + (a_{10}^2/4)\}/c^2\,.$$

Then

$$p(x) = x^7 + a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$

decomposes into the product of a cubic and a quartic with real coefficients as shown in Theorem A.

6. Numerical example

Consider the septic equation given below for solving by the proposed method:

$$x^7 - 10x^6 - 34x^5 + 440x^4 + 209x^3 - 4630x^2 + 2344x + 6720 = 0\,.$$

To know whether the above equation is solvable by the method described here, one has to check for the common root of the quintic (27) and the quartic (28). If there exists a common root, then we conclude that the above septic is solvable using the method given. For this purpose, the parameters, $a_7, a_8, a_9, a_{10}, a_{11}$, and a_{12} are determined as:

$$a_7 = -29.5\,,\, a_8 = 72.5\,,\, a_9 = 31.875\,,\, a_{10} = 10\,,\, a_{11} = 1031.625\,,\, a_{12} = -2098.125\,.$$

Using these parameters, the quintic (27) and the quartic (28) are given by:

$$c_2^5 + 35.75c_2^4 + 63.75c_2^3 - 1116c_2^2 + 284.7656 = 0$$
$$c_2^4 + 29c_2^3 + 57c_2^2 - 839.25c_2 - 430.3125 = 0\,.$$

Note that in the above quartic equation coefficient of c_2^4 is normalized to unity. Solving this quartic equation using the method given in [4], its roots are determined as: 4.5, -25.5, 1.714289, and -9.714292. For $c_2 = 4.5$, the quintic equation (27) is satisfied, which means, $c_2 = 4.5$, is the common root of (27) and (28). Therefore we conclude that the example septic equation is solvable with this method. The remaining unknowns, b_0, b_1, b_2, b_3, c_0, and c_1 are determined from equations (19), (18), (17), (16), (23) and (24) as:

$$b_0 = 42, b_1 = 72.5, b_2 = -29.5, b_3 = -5, c_0 = 42, c_1 = -7.5\,.$$

The quartic and the cubic equations, (30) and (31), which are the factors of the septic equation are given by:

$$x^4 - 5x^3 - 25x^2 + 65x + 84 = 0\,,$$

and,

$$x^3 - 5x^2 - 34x + 80 = 0\,.$$

Solving above quartic and the cubic equations [3, 4], their roots are determined as: -1, 3, -4, 7, and 2, -5, 8, respectively. Thus we have determined all the seven roots of the example septic equation.

7. Conclusions

We have presented a method for solving the septic equations in radicals using the so-called supplementary equation. The supplementary equation chosen decides the type of the solvable septic equation. The behavior of roots and the conditions to be met by the coefficients of such solvable septics are given.

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